Math 320 Midterm 2 Review Sheet
November 13, 2003

The exam covers everything up to the end of Chapter 5. The emphasis will be on the material we have discussed since the first midterm, namely sequences, limits, and continuity. Here is a sample syllabus:

1. **Topology on $\mathbb{R}$**

   open and closed sets; interior, closure and boundary of a set; compact sets; Bolzano-Weierstrass property; a subset of $\mathbb{R}$ is compact iff it has the Bolzano-Weierstrass property iff it is bounded and closed; nested set property of compact sets.

2. **Sequences**

   convergence of a sequence; convergent sequences are bounded; algebraic operations $+, -, \times, \div$ on convergent sequences respect limits; sandwich lemma; an increasing (resp. decreasing) sequence which is bounded above (resp. below) is convergent; definition of $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$; Cauchy sequences; a sequence in $\mathbb{R}$ is convergent iff it is Cauchy; subsequences; $\{x_n\}$ converges to L iff every subsequence of $\{x_n\}$ converges to L; a bounded sequence has a convergent subsequence (a form of the Bolzano-Weierstrass Theorem); definition of lim sup and lim inf; $\liminf x_n \leq \limsup x_n$ and the two are equal iff $\{x_n\}$ converges.

3. **Continuity**

   limit of a function at a point; sequential criterion for the existence of limit; algebraic operations on functions respect limits; one-sided limits; definition of continuity at a point; f is continuous at p iff $\lim_{n\to\infty} f(x_n) = f(p)$ for every sequence $\{x_n\}$ in the domain of f that converges to p; algebraic operations on functions respect continuity at a point; a composition of two continuous functions is continuous; the image of a compact set under a continuous function is compact; a continuous function on a compact set assumes its maximum and minimum; continuous functions have the intermediate value property; uniform continuity; a continuous function on a compact set is uniformly continuous.

Some practice problems

Problem 1. Let $S \subset \mathbb{R}$ be non-empty.

(i) Show that $p \in \mathrm{cl}(S)$ if and only if there is a sequence $\{x_n\}$ of points in S which converges to p.

(ii) Show that S is closed if and only if for every convergent sequence $\{x_n\}$ of points in S, $p = \lim_{n\to\infty} x_n$ is also in S.

Problem 2. Given a sequence $\{x_n\}$ of real numbers, suppose that the subsequences $\{x_{2k}\}$ and $\{x_{2k-1}\}$ both converge to L. Show that $\lim_{n\to\infty} x_n = L$.

Problem 3. Let $\{x_n\}$ be the sequence defined by

$$x_n = \begin{cases} 1 - \dfrac{1}{n} & \text{if } n = 3k \\ 2 & \text{if } n = 3k+1 \\ n + \dfrac{1}{n} & \text{if } n = 3k+2 \end{cases}$$

Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

Problem 4. Suppose $\lim_{x\to 0} \dfrac{f(x)}{x} = 2$ and f is continuous at 0. What is $f(0)$?

Problem 5. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Using the ε-δ definition of continuity, show that f is continuous at 0. Using the sequential criterion, verify that f does not have a limit at any $p \neq 0$, and conclude that f is not continuous at any $p \neq 0$.

Problem 6. Give an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ which has no maximum or minimum. (Recall that "bounded" means there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.)

Problem 7. Let $f : D \to \mathbb{R}$ be continuous at $p \in D$. If $f(p) > \lambda$, show that there is a $\delta > 0$ such that $f(x) > \lambda$ whenever $x \in D$ and $|x - p| < \delta$.