Math 320 Second Midterm Solutions

Problem 1.

(i) Define what it means for a set $S \subset \mathbb{R}$ to be **compact**. Then state the **Heine-Borel Theorem**.

S is compact if every open cover of S has a finite subcover. The Heine-Borel Theorem states that S is compact if and only if S is bounded and closed.

(ii) Define what it means for a sequence $\{x_n\}$ of real numbers to be a **Cauchy sequence**.

$\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer n_0 such that

$$|x_n - x_m| < \varepsilon \quad \text{whenever } n, m \geq n_0.$$

Problem 2.

(i) Let $\{x_n\}$ be a sequence of positive real numbers. Define a sequence $\{y_n\}$ by

$$y_n = \min_{1 \leq k \leq n} x_k.$$

Show that $\{y_n\}$ is convergent.

We have

$$y_1 = x_1 \quad y_2 = \min\{x_1, x_2\} \leq y_1 \quad y_3 = \min\{x_1, x_2, x_3\} \leq y_2 \quad \ldots$$

Hence $\{y_n\}$ is a decreasing sequence. On the other hand, $x_n > 0$ for all n, so $y_n > 0$ for all n. Hence $\{y_n\}$ is bounded below. It follows that $\{y_n\}$ is convergent.

(ii) Find a sequence $\{x_n\}$ of **integers** such that

$$\liminf_{n\to\infty} x_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} x_n = +\infty$$

Take, for example, the sequence

$$\{0, 1, 0, 2, 0, 3, 0, 4, \ldots\}$$

which can formally be defined as

$$x_n = \begin{cases} 0 & \text{if } n = 2k - 1 \\ k & \text{if } n = 2k \end{cases}$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$|f(x)| \le 2\sqrt{|x-3|} \qquad \text{for all } x \in \mathbb{R}.$$

Using the ε-δ definition of continuity, show that f is continuous at 3.

Setting $x = 3$ in the above inequality, we obtain $|f(3)| \le 0$, which gives $f(3) = 0$. Therefore, to prove continuity at 3, we must check that for every $\varepsilon > 0$ there is a δ such that $|f(x)| < \varepsilon$ whenever $|x-3| < \delta$.

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon^2}{4} > 0$. Then, if $|x-3| < \delta$, we have

$$|f(x)| \le 2\sqrt{|x-3|} < 2\sqrt{\delta} = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Problem 4. Are the following statements true or false? Justify your answer.

(i) There are compact sets A, B in $\mathbb{R}$ such that $A \cup B = \mathbb{R}$.

FALSE: Compact subsets of $\mathbb{R}$ are bounded, so if A and B are compact, then there are positive numbers M_1 and M_2 such that $A \subset [-M_1, M_1]$ and $B \subset [-M_2, M_2]$. Setting $M = \max\{M_1, M_2\}$, it follows that $A \cup B \subset [-M, M]$. Hence $A \cup B$ is bounded also. In particular, it cannot be the whole $\mathbb{R}$.

(ii) The equation $\cos x = x^2$ has at least two real solutions.

TRUE: Define $f(x) = \cos x - x^2$. Then f is continuous on $\mathbb{R}$ and the solutions of the above equation are the points where f vanishes. Note that

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi^2}{4} < 0 \qquad f(0) = 1 > 0 \qquad f\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} < 0$$

By the Intermediate Value Theorem, f must have a zero in $(-\frac{\pi}{2}, 0)$ and a zero in $(0, \frac{\pi}{2})$.

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $\{x \in \mathbb{R} : f(x) = 0\}$ is a closed set.

TRUE: Let $S = \{x \in \mathbb{R} : f(x) = 0\}$. To show S is closed, it suffices to check that whenever $\{x_n\}$ is a sequence in S which converges to some p, then $p \in S$. But this is almost clear: Suppose $x_n \in S$ for all n and $x_n \to p$. Then, by continuity, $f(p) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$, so $p \in S$.

Alternatively, we can show $\mathbb{R} \smallsetminus S$ is open. Let $p \in \mathbb{R} \smallsetminus S$. Then $f(p) \neq 0$ so either $f(p) > 0$ or $f(p) < 0$. Since f is continuous at p, there is a neighborhood $N(p, r)$ such that $f(x) > 0$ for all $x \in N(p, r)$ in the first case and $f(x) < 0$ for all $x \in N(p, r)$ in the second case. It follows that $N(p, r) \subset \mathbb{R} \smallsetminus S$. This proves that every point of $\mathbb{R} \smallsetminus S$ is an interior point.