MAT 127 Final II (Practice Exam)

Last Name: ______________ First Name: ______________ Student ID: ______________

Problem	1	2	3	4	5	6	**Total**
Points	**10**	**30**	**20**	**10**	**20**	**10**	**100**
Scores							

This midterm has five problems, weighted as shown. Please show your work – full credit may not be given if only the answers appear. **No calculators or books will be allowed on this test.** When calculating indefinite integrals, the answers should be in explicit forms, unless otherwise stated.

1. Determine whether the sequence converges or diverges, if it converges, find the limit.

$$a_n = e^{1/n}.$$

$\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1$

hence the sequence converges, and limit $= 1$.

2. Determine whether the series is convergent or divergent.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

Compare to $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$, Harmonic Series. diverges.

hence, since $\frac{n^2}{n^3+1} > \frac{n^2}{n^3} \Rightarrow$ $(\forall n)$

$\sum \frac{n^2}{n^3+1} > \sum \frac{1}{n}$

since $\sum \frac{1}{n}$ diverges.

So $\sum \frac{n^2}{n^3+1}$ diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$$

Use comparison test.

since $(-1)^n \le 1 \quad \forall n.$

So $\quad 2+(-1)^n \le 2+1=3 \quad \forall n.$

so $\sum \frac{2+(-1)^n}{n\sqrt{n}} \le \sum \frac{3}{n^{3/2}} = 3 \cdot \sum \frac{1}{n^{3/2}}$

since $\sum \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$

so it converges

therefore $\sum \frac{2+(-1)^n}{n\sqrt{n}}$ converges.

(c)

$$\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$$

Compare with $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ harmonic series div.

Want $\frac{n^2-5n}{n^3+n+1} > \frac{1}{n}$

$\Leftrightarrow n^3 - 5n^2 > n^3 + n + 1$ doesn't work.

Try Limit Comparison Test.

$$\lim_{n\to\infty} \frac{1}{n} \Big/ \frac{n^2-5n}{n^3+n+1} = \lim_{n\to\infty} \frac{n^3+n+1}{n(n^2-5n)}$$

$$= \lim_{n\to\infty} \frac{n^3+n+1}{n^3-5n^2}$$

$$= \lim_{n\to\infty} \frac{1+\cancel{\frac{1}{n^2}}+\cancel{\frac{1}{n^3}}}{1-\cancel{\frac{5}{n}}}$$

$$= 1 \neq 0 \text{ finite}$$

So by Limit Comp Test. since $\sum \frac{1}{n}$ div.

$\sum \frac{n^2-5n}{n^3+n+1}$ div.

3. Determine whether the series is absolute convergent, convergent, or divergent.

(a)

$$\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$$

Alternating Series Test:

check $\lim_{n\to\infty} b_n$: $b_n = \frac{3n-1}{2n+1}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$ doesn't satisfy.

Use Test for divergence:

$\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1} = \lim_{n\to\infty} (-1)^n \cdot \frac{3}{2}$ DNE

so the series diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

Ratio Test:

$a_n = 10^n / (n+1) 4^{2n+1}$

$$a_{n+1} = \frac{10^{n+1}}{(n+2)4^{2n+3}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{10^{\cancel{n+1}}}{(n+2)\underbrace{\cancel{4^{2n+3}}}_{4^2}} \Big/ \frac{\cancel{10^n}}{(n+1)\cancel{4^{2n+1}}}\right|$$

$$= \left|\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{10}{16} \cdot \frac{n+1}{n+2}\right| = \frac{10}{16} < 1$$

So by Ratio Test, the series converges.

4. Find the radius of convergence and the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$$

Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(4x+1)^{n+1}}{(n+1)^2} \Big/ \frac{(4x+1)^n}{n^2}\right|$$

$$= \frac{n^2}{(n+1)^2} \cdot |4x+1|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} |4x+1| = |4x+1|$$

① $|4x+1| < 1 \Rightarrow -1 < 4x+1 < 1 \Rightarrow -2 < 4x < 0$

$\Rightarrow -\frac{1}{2} < x < 0$ conv.

② $|4x+1| > 1 \Rightarrow x > 0$ or $x < -\frac{1}{2}$ div.

③ $|4x+1| = 1 \Rightarrow x = 0$ or $x = -\frac{1}{2}$ discuss:

if $x = 0$: $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2$ conv.

if $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ conv by Alt series Test.

Hence Int of conv: $-\frac{1}{2} \leq x \leq 0$

or $x \in [-\frac{1}{2}, 0]$

Radius $= \frac{0-(-\frac{1}{2})}{2} = \frac{1}{4}$.

5. Find a power series representation centered at 0 for the following functions

(a)

$$f(x) = \frac{1+x}{1-x}$$

$$f(x) = \frac{1+x}{1-x} = \frac{1}{1-x} + x \cdot \frac{1}{1-x}$$

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

So $\frac{x}{1-x} = x \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n$

So $f(x) = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} x^n$

$$= (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x + 2x^2 + 2x^3 + \cdots$$

$$= 1 + 2 \cdot \sum_{n=1}^{\infty} x^n$$

(b)

$$f(x) = \frac{1+x}{(1-x)^2}$$

$$f(x) = \frac{1}{(1-x)^2} + x \cdot \frac{1}{(1-x)^2}$$

find out $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)'$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (x^n)'$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{x}{(1-x)^2} = x \cdot \sum_{n=1}^{\infty} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n x^n$$

so $f(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1} + \sum_{n=1}^{\infty} n x^n$

$$= \sum_{n=1}^{\infty} n \cdot x^{n-1} + \sum_{n=2}^{\infty} (n-1) \cdot x^{n-1}$$

$$= 1 \cdot x^0 + \sum_{n=2}^{\infty} (n+(n-1)) \cdot x^{n-1}$$

$$= 1 + \sum_{n=2}^{\infty} (2n-1)\, x^{n-1}$$

$$= 1 + \sum_{n=1}^{\infty} (2n+1)\, x^n$$

6. Find the Taylor series for $f(x) = e^x$ centered at $a = 3$.

$$f(x) = e^x$$

$$f^{(1)}(x) = e^x$$

$$f^{(2)}(x) = e^x$$

$$\vdots$$

$$f^{(n)}(x) = e^x$$

so $f^{(n)}(3) = e^3 \quad \forall n$

so $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!} \cdot (x-3)^n$