MAT 127 Final I, Practice Exam

Last Name: _______________ First Name: _______________ Student ID: _______________

Problem	1	2	3	4	5	6	**Total**
Points	**10**	**10**	**40**	**10**	**10**	**20**	**100**
Scores							

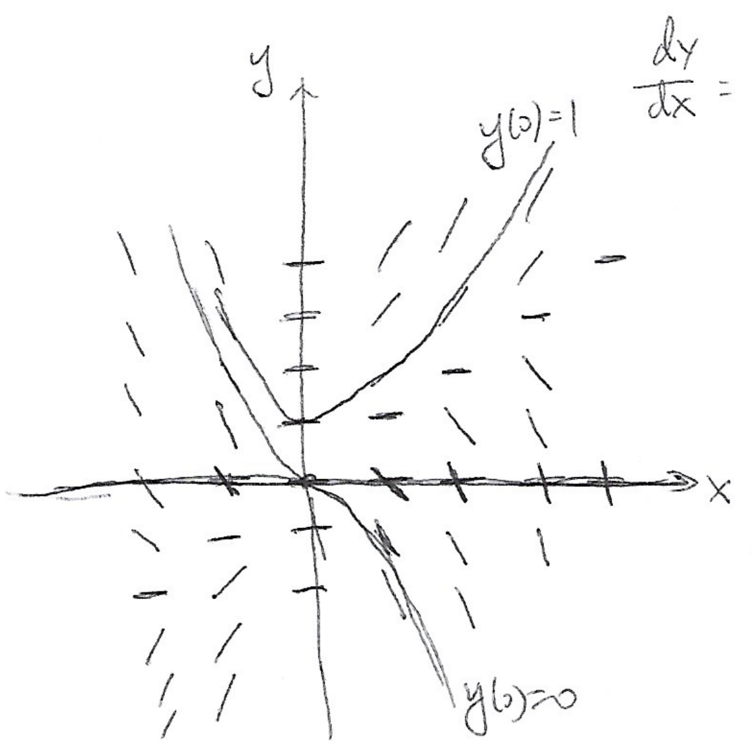
This midterm has five problems, weighted as shown. Please show your work – full credit may not be given if only the answers appear. **No calculators or books will be allowed on this test.** When calculating indefinite integrals, the answers should be in explicit forms, unless otherwise stated.

1. Sketch the directional field of the differential equation

$$\frac{dy}{dx} = xy - x^2.$$

Sketch the solution curves that satisfies the initial conditions: i) $y(0) = 0$; ii) $y(0) = 1$.

$$\frac{dy}{dx} = x(y-x)$$



2. Use Euler's Method with step size 0.1 to estimate $y(0.4)$, where y is the solution of the following initial-value problem

$$\frac{dy}{dx} = x + y^2, \qquad y(0) = 0.$$

$y_0 = 0 \qquad x_0 = 0$

$y(0.1) \approx y_1 = y_0 + 0.1 \cdot (x_0 + y_0^2) = 0 + 0.1(0 + 0^2) = 0$

$x_1 = 0.1$

$y(0.2) \approx y_2 = y_1 + 0.1(x_1 + y_1^2) = 0 + 0.1(0.1 + 0) = 0.01$

$x_2 = 0.2$

$y(0.3) \approx y_3 = y_2 + 0.1(x_2 + y_2^2) = 0.01 + 0.1(0.2 + (0.01)^2)$

$= 0.01 + 0.1(0.2001)$

$= 0.03001$

$x_3 = 0.3$

$y(0.4) \approx y_4 = y_3 + 0.1(x_3 + y_3^2)$

$= 0.03001 + 0.1(0.3 + (0.03001)^2)$

$= 0.0601 0006$

3. Solve the following **separable** differential equations:

(a) $\dfrac{dx}{dt} = 1 - t + x - tx.$

$$\frac{dx}{dt} = (1+x) - t(1+x)$$

$$= (1-t)(1+x)$$

$$\Rightarrow \quad \frac{1}{1+x}\,dx = (1-t)\,dt$$

$$\Rightarrow \quad \int \frac{1}{1+x}\,dx = \int (1-t)\,dt$$

$$\Rightarrow \quad \ln|1+x| = t - \frac{1}{2}t^2 + C$$

$$\Rightarrow \quad |1+x| = e^{t-\frac{1}{2}t^2+C} = A\cdot e^{t-\frac{1}{2}t^2}$$

$$\Rightarrow \quad x = Ae^{t-\frac{1}{2}t^2} - 1$$

(b) $\dfrac{dy}{dt} = \dfrac{e^y \sin^2 t}{y \sec t}$

$$\frac{y}{e^y}\,dy = \frac{\sin^2 t}{\sec t}\,dt$$

$$\Rightarrow \int \frac{y}{e^y}\,dy = \int \frac{\sin^2 t}{\sec t}\,dt$$

LHS: $\displaystyle\int \frac{y}{e^y}\,dy = \int y \cdot e^{-y}\,dy$

Int by Parts: $u = y$ $\quad du = dy$

$dv = e^{-y}dy$ $\quad v = -e^{-y}$

$$\int \frac{y}{e^y}\,dy = \int y\cdot e^{-y}dy = -y\cdot e^{-y} + \int e^{-y}dy$$

$$= -y\cdot e^{-y} - e^{-y}$$

RHS: $\displaystyle\int \frac{\sin^2 t}{\sec t}\,dt$, since $\sec t = \dfrac{1}{\cos t}$, $\dfrac{\sin^2 t}{\sec t} = \sin^2 t \cdot \cos t$

$$\int \sin^2 t \cos t\,dt$$

substitution: let $u = \sin t$

$du = \cos t\,dt$

$$\int \sin^2 t\,(\cos t\,dt) = \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\sin^3 t$$

So solution: $-ye^{-y} - e^{-y} = \frac{1}{3}\sin^3 t + C.$

(c) $\frac{dr}{dt} + 2tr = r, \qquad r(0) = 5$

$$\frac{dr}{dt} = r - 2tr$$
$$= r(1-2t)$$

$$\Rightarrow \frac{1}{r}\, dr = (1-2t)\, dt$$

$$\Rightarrow \int \frac{1}{r}\, dr = \int (1-2t)\, dt$$

$$\Rightarrow \ln|r| = t - t^2 + C$$

$$\Rightarrow r = e^{t-t^2+C} = A \cdot e^{t-t^2}$$

$$5 = r(0) = A \cdot e^{0-0}$$
$$\Rightarrow A = 5$$
$$\Rightarrow r(t) = 5 \cdot e^{t-t^2}$$

(d) $x \ln x = y(1+\sqrt{3+y^2})y', \qquad y(1) = 1.$

$$x\ln x = y\left(1+\sqrt{3+y^2}\right)\frac{dy}{dx}$$

$$\Rightarrow \int y\left(1+\sqrt{3+y^2}\right)dy = \int x\ln x \, dx$$

<u>LHS</u>: let $3+y^2 = u \Rightarrow du = 2y\,dy$

$$\Rightarrow \frac{1}{2}du = y\,dy$$

$$\begin{aligned} \text{LHS} &= \int (1+\sqrt{u})\cdot\frac{1}{2}du \\ &= \frac{1}{2}\int \left(1+u^{\frac{1}{2}}\right)du \\ &= \frac{1}{2}u + \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} \\ &= \frac{1}{2}u + \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{2}(3+y^2) + \frac{1}{3}(3+y^2)^{\frac{3}{2}} \end{aligned}$$

<u>RHS</u>: <u>Int by Parts</u>: $\quad u = \ln x \qquad du = \frac{1}{x}dx$

$$dv = x\,dx \qquad v = \frac{1}{2}x^2$$

$$\begin{aligned} \int x\ln x\,dx &= \frac{1}{2}x^2\cdot\ln x - \int \frac{1}{2}x^{\cancel{2}}\cdot\frac{1}{\cancel{x}}dx \\ &= \frac{1}{2}x^2\cdot\ln x - \int\frac{1}{2}x\,dx \\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \end{aligned}$$

so solution: $\frac{1}{2}(3+y^2) + \frac{1}{3}(3+y^2)^{\frac{3}{2}} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

and $y(1)=1 \Rightarrow$ when $x=1$, $y=1 \Rightarrow$

$$\frac{1}{2}(3+1) + \frac{1}{3}(3+1)^{\frac{3}{2}} = \frac{1}{2}\cdot 1^2\cdot\ln 1 - \frac{1}{4}\cdot 1^2 + C$$

$$\Rightarrow 2 + \frac{1}{3}\cdot 8 = 0 - \frac{1}{4} + C$$

$$\Rightarrow C = \frac{14}{3} + \frac{1}{4} = \frac{59}{12}$$

$$\boxed{\frac{1}{2}(3+y^2) + \frac{1}{3}(3+y^2)^{\frac{3}{2}} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{59}{12}}$$

4. Find the orthogonal trajectories of the family of curves: $y = e^{kx}$.

Step 1: $\frac{dy}{dx} = ke^{kx}$

Step 2: Find k: $\ln y = kx \Rightarrow k = \frac{\ln y}{x}$

So $\frac{dy}{dx} = \frac{\ln y}{x} \cdot e^{\frac{\ln y}{x} \cdot x}$

$= \frac{\ln y}{x} \cdot e^{\ln y} = \frac{y}{x} \cdot \ln y \quad (= m_1)$

Step 3: ~~xxxx~~ DE for Ortho. traj:

$$\frac{dy}{dx} = -\frac{1}{m_1} = -\frac{x}{y \ln y}$$

Step 4: solve: $(y \ln y)\, dy = -x\, dx$

$\Rightarrow \int y \ln y\, dy = -\int x\, dx$

LHS: Int by Parts: $u = \ln y \qquad du = \frac{1}{y} dy$

$dv = y\, dy \qquad v = \frac{1}{2} y^2$

$\int y \ln y\, dy = \frac{1}{2} y^2 \ln y - \frac{1}{2} \int \frac{y^2}{y} dy$

$= \frac{1}{2} y^2 \ln y - \frac{1}{4} y^2$

RHS: $-\int x\, dx = -\frac{1}{2} x^2 + C$

Ans: $\frac{1}{2} y^2 \ln y - \frac{1}{4} y^2 = -\frac{1}{2} x^2 + C.$

5. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample. Find the mass that remains after t years.

let $m(t)$ be the mass of cesium-137 after t-years

then $\frac{dm(t)}{dt} = k\, m(t)$

$$\Rightarrow m(t) = m(0) \cdot e^{kt}$$

$m(0) = 100$ (mg)

$$\Rightarrow m(t) = 100 \cdot e^{kt}$$

Use half-life to find k: when $t = 30$, $m(30) = \frac{1}{2} \cdot 100$
$= 50$

$$\Rightarrow 100 \cdot e^{30t} = 50$$

$$\Rightarrow e^{30t} = \frac{1}{2} \Rightarrow 30t = \ln\frac{1}{2} \Rightarrow t = \frac{1}{30} \cdot \ln\frac{1}{2}$$

$$\Rightarrow m(t) = 100 \cdot e^{\frac{1}{30}\ln\frac{1}{2} \cdot t}$$

$$= 100 \cdot e^{-\ln 2 \cdot \frac{t}{30}} \quad \cancel{= 100 \cdot e}$$

$$= 100 \cdot 2^{-\frac{t}{30}}$$

6. Solve the following **second order** differential equations.

(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 4y = 0.$

Auxiliary Equation : $r^2 - 6r + 4 = 0$

$b^2 - 4ac = 36 - 4 \cdot 4 = 20 > 0$

so $r_{1,2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5} \Rightarrow r_1 = 3 + \sqrt{5}$

$r_2 = 3 - \sqrt{5}$

thus : $y = A \cdot e^{(3+\sqrt{5})x} + B \cdot e^{(3-\sqrt{5})x}$

(b) $y'' + 16y = 0, \qquad y(\pi/4) = -3, \qquad y'(\pi/4) = 4.$

Auxiliary Equation: $r^2+16=0$

$b^2-4ac = 0-4\cdot 16 = -64 < 0$

$r_{1,2} = \frac{-b \pm i\sqrt{4ac-b^2}}{2a} = \frac{\cancel{0} \pm i\sqrt{64}}{2}$

$= \pm 4i$

$\Rightarrow \alpha = 0 \quad \beta = 4$

solution: $\cancel{y = e^{0x}(C_1 \cos 4\beta + C_2 \sin 4\beta)}$

$\cancel{= C_1 \cos 4\beta}$

$y = e^{0x}(A\cdot\cos(4x) + B\cdot\sin(4x))$

$= A\cos(4x) + B\sin(4x)$

$x = \frac{\pi}{4} \quad y\left(\frac{\pi}{4}\right) = A\cdot\cos\pi + B\cdot\sin\pi$

$= -A = -3 \Rightarrow A = 3$

$y'(x) = -4A\cdot\sin(4x) + 4B\cdot\cos(4x)$

$y'\left(\frac{\pi}{4}\right) = -4A\sin\pi + 4B\cos\pi$

$= 4B = 4 \quad \Rightarrow B = 1$

<u>Ans</u>: $y = 3\cos 4x + \sin 4x.$