MAT561 Homework 2

Due Wednesday, March 26th.

Abstract

As usual, you may not skip any exercises and your solutions must show that you have understood the solution to the problem.

1 Superspace conventions

In class I tried to keep Frankel's [2] conventions and really screwed the whole thing up. In this exercise we will work through a consistent set of conventions I lifted from the very clear and careful book [1] by Buchbinder and Kuzenko. Their choice is very clever: They use "mostly-plus" conventions for the Minkowski metric $\eta = \mathrm{diag}(-1, +1, +1, +1)$ which is the popular choice for geometers and relativists but they define the Clifford algebra relation with an extra sign, that is, they take $\{\gamma^a, \gamma^b\} = -2\eta^{ab}\mathbf{1}$. The result is that in the discussion of spinors the conventions closely resemble the "mostly-minus" choice favored by practitioners of quantum field theory and pretty much every book ever written on the subject and the only cost is an extra factor of $i = \sqrt{-1}$ in front of all your Dirac matrices (and of course all consequences thereof).

A left-handed Weyl spinor representation (also known as the $\mathbf{2}$)will be taken to have its $SL(2, \mathbb{C})$ index down ψ_α. The "metric" tensor $\varepsilon^{\alpha\beta}$ is normalized by $\varepsilon^{12} = +1$ and its inverse is defined by $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$. An index on a left-handed Weyl spinor is raised according to the convention $\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta$.[1] Show that this implies that $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$. The complex conjugate representation $\bar{\mathbf{2}}$ has index structure given by $\chi_{\dot\alpha}$. Indices are raised and lowered with the conjugate $\varepsilon_{\dot\alpha\dot\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$ with

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[1] Note that there is no factor of i here contrary to the conventions in class.

the same conventions as above. Define the pairing of co- and contravariant spinors ψ_α and χ^α and their conjugates by $\langle\chi\psi\rangle \equiv \chi^\alpha\psi_\alpha$ and $[\bar\chi\bar\psi] \equiv \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha}$. Compute $\langle\psi\chi\rangle$ and $[\bar\psi\bar\chi]$ in terms of these. We usually drop the bracket notation when there can be no confusion. Show that $\psi_\alpha\chi_\beta = \psi_\beta\chi_\alpha + \varepsilon_{\alpha\beta}\psi\chi$ and $\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = \bar\psi_{\dot\beta}\bar\chi_{\dot\alpha} - \varepsilon_{\dot\alpha\dot\beta}\bar\psi\bar\chi$.

The Pauli matrices are extended to a 4-covector (note the placement of the spacetime index!) of matrices by $(\sigma_a) = (\mathbf{1}, \vec\sigma)$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and their index structure is defined to be $(\sigma_a)_{\alpha\dot\alpha}$. Define the associated matrices $(\tilde\sigma_a)^{\dot\alpha\alpha} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma_a)_{\beta\dot\beta}$ and show that $(\tilde\sigma_a) = (\mathbf{1}, -\vec\sigma)$. These index structures define a natural matrix multiplication *exempli gratia* $(\sigma^a\tilde\sigma^b)_\alpha{}^\beta = (\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma^b)^{\dot\alpha\beta}$. Verify the formulæ

$$\begin{aligned}
(\sigma^a\tilde\sigma^b + \sigma^b\tilde\sigma^a)_\alpha{}^\beta &= -2\eta^{ab}\delta_\alpha^\beta \\
(\tilde\sigma^a\sigma^b + \tilde\sigma^b\sigma^a)^{\dot\alpha}{}_{\dot\beta} &= -2\eta^{ab}\delta_{\dot\beta}^{\dot\alpha} \\
(\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} &= -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}
\end{aligned} \tag{2}$$

Define the combinations $(\sigma^{ab})_\alpha{}^\beta \equiv -\frac{1}{4}(\sigma^a\tilde\sigma^b - \sigma^b\tilde\sigma^a)_\alpha{}^\beta$ and $(\tilde\sigma^{ab})^{\dot\alpha}{}_{\dot\beta} \equiv -\frac{1}{4}(\tilde\sigma^a\sigma^b - \tilde\sigma^b\sigma^a)^{\dot\alpha}{}_{\dot\beta}$. Show that $(\sigma^{ab})_{\alpha\beta} = (\sigma^{ab})_{\beta\alpha}$ and similarly for $\tilde\sigma^{ab}$. Furthermore, verify that as 2-forms, σ_{ab} is self-dual and $\tilde\sigma_{ab}$ is anti-self-dual. Show that this, together with the equations above, means that a symmetric spin-tensor $S_{\alpha\beta} = S_{\beta\alpha}$ is equivalent to a self-dual 2-form and analogously for $S'_{\dot\alpha\dot\beta}$. Take the results you have derived so far to conclude that in all we have that an anti-symmetric spin-tensor $A_{\alpha\beta}$ is a scalar, a symmetric one ($S'_{\dot\alpha\dot\beta}$) $S_{\alpha\beta}$ is a(n anti-)self-dual 2-form, and one with mixed indices $V_{\alpha\dot\alpha}$ is a vector. In terms of irreducible representations of the 4-dimensional Lorentz group these statements are condensed into the decomposition rules

$$\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{3} \;,\; \bar{\mathbf{2}}\otimes\bar{\mathbf{2}} = \mathbf{1}\oplus\bar{\mathbf{3}} \;,\; \text{and } \mathbf{2}\otimes\bar{\mathbf{2}} = \mathbf{4} \tag{3}$$

where $\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{4}$ are, respectively, the scalar, spinor, conjugate spinor, self-dual tensor, anti-self-dual tensor, and vector (a.k.a. defining) representations labelled by their dimensions. Note that the adjoint $\mathbf{6} = \mathbf{3}\oplus\bar{\mathbf{3}}$ is irreducible as a real representation.

Show that the matrices

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \tilde\sigma^a & 0 \end{pmatrix} \tag{4}$$

satisfy the Clifford algebra relations (with the extra sign). Explain why this implies that a Dirac spinor must have the form

$$(\Psi_{\hat\alpha}) = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} . \tag{5}$$

The rotation generators are defined by $\Sigma^{ab} = -\frac{1}{4}[\gamma^a, \gamma^b]$ and a Dirac spinor is defined to transform as $\delta\Psi = \frac{1}{2}\omega_{ab}\Sigma^{ab}\Psi$.

Moving on to superspace, we take $\mathbb{R}^{4|4}$ to have coordinates $(x^A) = (x^a, \theta_\alpha, \bar\theta^{\dot\alpha})$. Differentiation proceeds as defined in class. Conjugation, however, is taken to reverse the order of monomials. For example $(\theta^\alpha\theta^\beta)^* = \bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}$. Go back and look at the definition of the contraction of dotted spinors. Show that $\langle\psi\chi\rangle^* = [\bar\chi\bar\psi]$. In particular note that $(\theta^2)^* = \bar\theta^2$. Carefully analyzing what this implies for derivatives, one finds that for a general superfunction f with parity $|f|$ the conjugate of the derivative is $(\partial_\alpha f)^* = -(-)^{|f|}\bar\partial_{\dot\alpha}\bar f$.

The global supersymmetry transformation with constant spinor parameter ϵ is defined to act on the coordinates as

$$\delta\theta^\alpha = \epsilon^\alpha \ , \ \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \ , \ \delta x^a = i(\theta\sigma^a\bar\epsilon - \epsilon\sigma^a\bar\theta) \ . \tag{6}$$

We can write this as $\delta x^A = -i(\epsilon Q + \bar\epsilon\bar Q)x^A$ with the supercharges defined by

$$Q_\alpha = i\partial_\alpha + (\sigma^a\bar\theta)_\alpha\partial_a \ , \ \bar Q_{\dot\alpha} = -i\bar\partial_{\dot\alpha} - (\theta\sigma^a)_{\dot\alpha}\partial_a \tag{7}$$

Show that the only non-vanishing bracket (ignoring rotations) is

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2(\sigma^a)_{\alpha\dot\alpha}P_a \tag{8}$$

where $P_a = -i\partial_a$. By taking into consideration that a scalar superfield $V(x,\theta,\bar\theta)$ is invariant under supersymmetry $V'(x',\theta',\bar\theta') = V(x,\theta,\bar\theta)$ (tensor transformation law) show that $\delta V(x,\theta,\bar\theta) = V'(x,\theta,\bar\theta) - V(x,\theta,\bar\theta)$ where by definition we keep only the linear part, is given by

$$\delta V(x,\theta,\bar\theta) = i(\epsilon Q + \bar\epsilon\bar Q)V(x,\theta,\bar\theta) \ . \tag{9}$$

Introduce the covariant derivatives

$$D_\alpha = \partial_\alpha + i(\sigma^a\bar\theta)_\alpha\partial_a \ , \ \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i(\theta\sigma^a)_{\dot\alpha}\partial_a \tag{10}$$

and show that they commute with the supercharges. Show that the only non-vanishing commutator of the Ds is

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^a)_{\alpha\dot\alpha}\partial_a \ . \tag{11}$$

Define the squares $D^2 = \langle DD \rangle$ and $\bar{D} = [\bar{D}\bar{D}]$ and show that[2]

$$[D^2, \bar{D}_{\dot\alpha}] = -4i\partial_{\alpha\dot\alpha} D^\alpha \text{ and } [\bar{D}^2, D_\alpha] = +4i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} \ . \tag{12}$$

Now show that

$$D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} \ . \tag{13}$$

Using these equations show that[3]

$$D^2\bar{D}^2 + \bar{D}^2 D^2 - 2D^\alpha \bar{D}^2 D_\alpha = 16\Box \tag{14}$$

and

$$[D^2, \bar{D}^2] = -4i\partial_{\alpha\dot\alpha}[D^\alpha, \bar{D}^{\dot\alpha}] \ . \tag{15}$$

2 Non-linear σ-model

If you do not know what a Kähler manifold is, find out. Consider the action of n chiral fields Φ^i and their anti-chiral conjugates $\bar\Phi^{\bar\imath}$

$$S[\Phi^i, \bar\Phi^{\bar\imath}] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K[\Phi^i, \bar\Phi^{\bar\imath}] \tag{16}$$

for K a general real function of Φ and $\bar\Phi$. Show that this action has as a symmetry any (holomorphic) field transformation

$$\Phi \mapsto f(\Phi) \tag{17}$$

which takes

$$K[\Phi, \bar\Phi] \mapsto K[\Phi, \bar\Phi] + \Lambda(\Phi) + \bar\Lambda(\bar\Phi) \ . \tag{18}$$

Note that $(\bar\Lambda)$ Λ is (anti-)chiral. Define the components

$$\varphi^i = \Phi^i| \ , \ \psi^i_\alpha = D_\alpha \Phi^i| \ , \ F^i = -\frac{1}{4}D^2\Phi^i| \tag{19}$$

and similarly for the conjugates. Find the component form of the action. In particular, put the kinetic term for the scalars in the the form $-\int \mathrm{d}^4x\, g_{i\bar\jmath}(\varphi, \bar\varphi)\, \partial^a\bar\varphi^{\bar\jmath}\partial_a\varphi^i$. A theory with such a scalar kinetic term is called a non-linear σ-model. Why is it appropriate to refer to K as the Kähler potential?

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[2] Hint: Start with $\bar{D}_{\dot\alpha} D^2$ and move the $\bar{D}$ to the right using the basic rule (11). Do the same for the other case.

[3] You shouldn't have to do much at this point but remember $D_\alpha D_\beta D_\gamma = 0$ identically.

Study the transformation of the component fields (19) under the holomorphic transformation (17). Observe that F is not transforming covariantly, that is, if we interpret φ^i as a coordinate and the transformation as a reparameterization, ψ^i is transforming as a tensor but F^i is not. Fix this by introducing the Christoffel symbols

$$\Gamma^I_{JK} = \frac{1}{2} g^{IL} \left(\partial_J g_{KL} + \partial_K g_{JL} - \partial_L g_{JK} \right) \tag{20}$$

and defining the combinations

$$\begin{aligned} \mathcal{F}^i &= F^i - \frac{1}{4} \Gamma^i_{jk} \, \langle \psi^j \psi^k \rangle \\ \bar{\mathcal{F}}^{\bar{\imath}} &= \bar{F}^{\bar{\imath}} - \frac{1}{4} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \, [\bar{\psi}^{\bar{\jmath}} \bar{\psi}^{\bar{k}}] \ . \end{aligned} \tag{21}$$

Check that the Christoffel symbols not used here all vanish. Show that the component action can be put into the form

$$\begin{aligned} S &= -\int \mathrm{d}^4 x \, g_{i\bar{\jmath}} \left(\partial^a \bar{\Phi}^{\bar{\jmath}} \partial_a \Phi^i - \bar{\mathcal{F}}^{\bar{\jmath}} \mathcal{F}^i + \frac{i}{4} \psi^i \sigma^a \overleftrightarrow{\nabla}_a \bar{\psi}^{\bar{\jmath}} \right) \\ & \qquad - \frac{1}{16} \int \mathrm{d}^4 x \, \mathcal{R}_{i\bar{\jmath}k\bar{l}} \, \psi^i \bar{\psi}^{\bar{\jmath}} \psi^k \bar{\psi}^{\bar{l}}, \end{aligned} \tag{22}$$

where we have introduced the covariant derivatives

$$\begin{aligned} \nabla_a \psi^i_\alpha &= \partial_a \psi^i_\alpha + \Gamma^i_{jl} (\partial_a \varphi^j) \psi^l_\alpha \\ \nabla_a \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} &= \partial_a \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} + \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{l}} (\partial_a \bar{\varphi}^{\bar{\jmath}}) \bar{\psi}^{\bar{l}}_{\dot{\alpha}} \end{aligned} \tag{23}$$

and $\mathcal{R}_{i\bar{\jmath}k\bar{l}} = K_{ik\bar{\jmath}\bar{l}} - g^{m\bar{n}} K_{ik\bar{n}} K_{m\bar{\jmath}\bar{l}}$ in terms of derivatives of the Kähler potential. Show that this is the $IJKL = i\bar{\jmath}k\bar{l}$ component of the Riemann tensor

$$\mathcal{R}^I{}_{JKL} = \partial_K \Gamma^I_{LJ} - \partial_L \Gamma^I_{KJ} + \Gamma^I_{KE} \Gamma^E_{LJ} - \Gamma^I_{LE} \Gamma^E_{KJ} \tag{24}$$

with the I index lowered.

References

[1] I. L. Buchbinder and S. M. Kuzenko, “Ideas and methods of supersymmetry and supergravity: Or a walk through superspace,” SPIRES entry *Bristol, UK: IOP (1998) 656 p*

[2] T. Frankel, “The geometry of physics: An introduction,” SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*