MAT561 Homework 1

Due Wednesday, March 5$^{\text{th}}$.

Abstract

As usual, you may not skip any exercises and your solutions must show that you have understood the solution to the problem. The last 2 problems use concepts from last semester's course. It is important for you to understand them so if you are having difficulty, please come see me.

1 Reading

Chapters 19 in Frankel [1].

2 Spinors in various dimensions

This problem is taken from the appendix of [2]. Consider the Clifford algebra relation

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}\mathbf{1} \tag{1}$$

where $a, b = 1, \ldots, d$ are d-dimensional spinor indices, η is the Minkowski metric in the "mostly plus" convention and the Γs and $\mathbf{1}$ are matrices in $GL(n, \mathbb{C})$ for some n. Suppose $d = 2k + 2$ is even[1] and group the matrices into $k + 1$ matrices

$$\gamma^{0\pm} = \frac{1}{2}\left(\pm\gamma^0 + \gamma^1\right), \gamma^{i\pm} = \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right), i = 1, \ldots, k. \tag{2}$$

[1] If d is odd, we proceed with the construction as if we were in $d-1$ dimensions. The missing Dirac matrix is then taken to be the chirality matrix (c.f. equation 6).

Compute their anti-commutators. Conclude that $(\gamma^{i-})^2 = 0$ for all $i = 0, \ldots, k$. This implies that in the representation space $\mathbb{C}^n$ there is a vector ζ which is annihilated by all these "lowering operators"

$$\gamma^{i-}\zeta = 0 \quad \forall i = 0, \ldots, k. \tag{3}$$

From this vector we may construct others $\zeta^{(\mathbf{s})}$, $\mathbf{s} = (s_0, \ldots, s_k)$ by acting in all ways with the "raising operators" γ^{i+}. Argue from the anti-commutation relations that we get an $n = 2^{k+1}$-dimensional representation this way. We could label these representations by 0s and 1s but let us instead shift this by $-\frac{1}{2}$ resulting in a labeling $\mathbf{s} = (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ and

$$\zeta^{(\mathbf{s})} = (\gamma^{k+})^{s_k+\frac{1}{2}} \ldots (\gamma^{0+})^{s_0+\frac{1}{2}}\zeta. \tag{4}$$

Starting in $k = 1$ and using $\zeta^{(\mathbf{s})}$ as a basis, derive the explicit form of the Dirac matrices Γ^0 and Γ^1. Do the same for $k = 2$.[2] Give an inductive prescription to generate an explicit set of Dirac matrices for general k.

We know that the Lorentz generators $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ satisfy the $\mathfrak{so}(d-1, 1)$ relations. Use this to show that the operators[3]

$$S^i \equiv i^{\delta_{i,0}-1}\Sigma^{2i,2i+1} \tag{5}$$

commute with each other. Show that the eigenvalues of these operators are the $\{s^i\}$.

Define the chirality matrix

$$\gamma \equiv i^{-k}\gamma^0\gamma^1 \ldots \gamma^{d-1} \tag{6}$$

and show that it anti-commutes with the Dirac matrices, commutes with the Lorentz generators, and squares to 1. By writing γ in terms of the S^i operators, show that $\gamma = +\mathbf{1}$ on spinors $\zeta^{(\mathbf{s})}$ with even numbers of $s^i = -\frac{1}{2}$ and $\gamma = -\mathbf{1}$ on those with an odd number of $s^i = -\frac{1}{2}$.

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[2] Due to the definition of $\Gamma^{a\pm}$ the basis you get will not agree with the basis used in class. The various explicit representations are equivalent (they are related by similarity transformations which preserve the Clifford algebra relations) but some have special properties. The *Weyl basis* used in class gave a nice block-diagonal form for the Lorentz generators. There is also the *Majorana basis* in which the Dirac matrices are all real. In the *Dirac basis* the representation of the physical degrees of freedom in a spinor is nice. In this problem Polchinski is using a basis (I don't know of a name but we can call it a Fock basis) which the spinor is built up by acting on the ground state by raising operators in a nice way.

[3] The annoying factor of $i^{\delta_{i,0}}$ is just there to remove the factor of $-i$ in S^0.

Argue from the construction of the Dirac matrices above that the irreducible spinor representations are unique up to a change of basis. Since the complex conjugate matrices $\pm(\gamma^a)^*$ satisfy the same Clifford algebra relations, they are related to the γs by a similarity transformation. Noting that in Polchinski's basis $\zeta^{(\mathbf{s})}$ the matrices $\gamma^{a\pm}$ are real, γ^a is imaginary for $a = 3, 5, 7, \ldots$. Define the matrices

$$B_1 = \gamma^3\gamma^5 \ldots \gamma^{d-1} \quad \text{and} \quad B_2 = \gamma B_1 \tag{7}$$

and show that

$$B_1\gamma^a B_1^{-1} = (-1)^k \gamma^{a*} \quad \text{and} \quad B_2\gamma^a B_2^{-1} = (-1)^{k+1}\gamma^{a*} \tag{8}$$

What does a similarity transformation of the Lorentz generators by these matrices do to the generators? Use this to construct from ζ^* and the Bs a spinor which transforms like ζ, that is, construct the charge conjugate of ζ. Square the charge conjugation operation to determine in which dimensions the Majorana condition can be imposed.

Compute the effect on the chirality matrix of a similarity transformation by the Bs. Use this to determine in which dimensions the Majorana condition can be imposed on a Weyl spinor. Such spinors are simply called Majorana-Weyl.

3 Solutions to the Dirac equation

This problem is from §3 of [3]. Suppose a Dirac spinor Ψ solves the massive Dirac equation. Recall that the components of Ψ satisfy the Klein-Gordon equation and argue that any such Ψ can be written as a linear combination of plane waves[4]

$$u(k)\mathrm{e}^{ik\cdot x} \tag{9}$$

with wave number k_a. What is the dispersion relation $k^2(m)$? Plug this *anzats* into the Dirac equation to get an algebraic equation. Now perform a standard trick: Argue physically or mathematically that if $k^0 > 0$, you can always go to a frame in which the wave number is $(k_a) = (-m, 0, 0, 0)$. Solve the Dirac equation for Ψ in this "rest frame" in terms of a 2-component spinor ξ.

Now boost this solution to a frame with rapidity η in the 3-direction: $(k_a) = (-m\coth\eta, 0, 0, m\sinh\eta)$. In this frame, a good

[4]This is Fourier's decomposition.

basis for the ξ is the σ^3 eigen-basis ξ^s with $s = 1, 2$ with

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{10}$$

Supposing a the boost is highly relativistic, what form does the Dirac spinor take for $s = 1$ and $s = 2$? Define the "helicity" operator

$$h = \frac{1}{2}\hat{k}^i \ell^i \ . \tag{11}$$

where $\hat{k}$ is the unit wave number indicating the direction of motion of the plane wave and the $\ell^i = \frac{1}{2}\epsilon^{ijk}\Sigma_{jk}$ are the spacial rotation generators. The helicity is the projection of the spinor's "angular momentum" or "spin" onto it's direction of motion. Argue that the helicity of a massless spinor is well-defined meaning that it does not depend on the reference frame.

The choices about reference frames made above may look *ad hoc* but, as usual by Lorentz covariance, any other frame is equivalent to this one. In particular we have found that any spinor with $k^0 > 0$ satisfying the Dirac equation describes 2 real degrees of freedom which, in the case of $m \neq 0$ we may think of as "spin up" and "spin down" along some spacial axis or or in the case $m = 0$ helicity $\pm\frac{1}{2}$.

4 Coupling to gauge fields

The Lagrangian of a free Dirac spinor is invariant under a global $U(1)$ action $\Psi \mapsto \mathrm{e}^{i\alpha}\Psi$. Gauge this global symmetry thereby constructing the coupling of Dirac spinors to gauge fields. What is the conserved current?

References

[1] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*

[2] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," SPIRES entry *Cambridge, UK: Univ. Pr. (1998) 531 p*

[3] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," SPIRES entry *Reading, USA: Addison-Wesley (1995) 842 p*