MAT560 Homework 5

Due Wednesday, December 19^{th}

1 Reading

Read §18 in Frankel.

2 Kaluza-Klein unification

In homework 4 you showed that the theory with Lagrangian

$$2\mathcal{L} \;=\; h^{ab}\Box h_{ab} - h\Box h - 2h^{ab}\partial_a\partial^c h_{bc} + 2h\partial^a\partial^b h_{ab} \tag{1}$$

is the unique kinematical theory for a symmetric rank-2 tensor with gauge invariance

$$\delta h_{ab} = \partial_a \xi_b + \partial_b \xi_a \ . \tag{2}$$

At no point in the analysis was the dimension of space-time relevant. Assuming we are in $\mathbb{R}^{4,1}$ amounts to replacing all indices a, b with new indices A, B such that the new indices take on both the old 4-dimensional values $A, B = a, b = 0, \ldots, 3$ together with a new value traditionally called $A, B = 5$. That is $(x^A) = (x^a, x^5) = (x^0, x^1, x^2, x^3, x^5)$.[1]

Replace all the 4-dimensional indices in the Lagrangian with 5-dimensional indices. Do the same for the gauge transformation of the metric. Next *assume* that none of the field components $h_{AB}(x)$ depend on x^5. Now reduce the 5-dimensional notation back to "4-dimensional plus an extra dimension" notation by writing out for example

$$h^{ab}\Box h_{ab} \;\;\rightarrow\;\; h^{AB}\hat{\Box} h_{AB}$$

[1]The value x^4 is skipped for historical reasons.

$$
\begin{aligned}
&= h^{ab}(\Box + \partial_5^2)h_{ab} + 2h^{a5}(\Box + \partial_5^2)h_{a5} + h^{55}(\Box + \partial_5^2)h_{55} \\
&= h^{ab}\Box h_{ab} + 2h^{a5}\Box h_{a5} + h^{55}\Box h_{55} \qquad (3)
\end{aligned}
$$

where the $\hat{\Box}$ is the 5-dimensional d'Alembertian and the $\Box$s in the last line are understood to be 4-dimensional.

We will show that the resulting 4-dimensional theory describes a linearized theory of gravity together with Maxwell theory with gauge field $A_a \propto h_{a5}$ and a real scalar $\varphi \propto h_{55}$. To do this show that the Lagrangian reduces to a Lagrangian for h_{ab} and a Maxwell Lagrangian for A_a.[2] What is the correct constant of proportionality for $A_a \propto h_{a5}$ which gives the proper normalization of the Maxwell Lagrangian? Does A_a have the proper gauge invariance?

The calculation we just described prompted Kaluza and Klein to put forward the following scenario for the unification of gravity and electromagnetism: They posed that we actually live in a 5-dimensional space-time but that for some reason the "extra dimension" is exceedingly small. Sufficiently so that we would not be able to observe it with the experiments performed so far. One can show that for the components of the gravitational field h_{AB} to penetrate into this miniscule extra dimension, the contribution $(\partial_5 h_{AB})^2$ to the energy of that component must be very large.[3] For energies small compared to the scale associated to the extra dimension, components constant in x^5 are heavily favored and to the first approximation we can ignore the non-constant (in x^5) parts. We have just seen then that gravity in such a spacetime looks like gravity+electromagnetism+scalar. A concrete realization of such a background would be a spacetime of the form $\mathbb{R}^{3,1}\times$

The scenario described above was abandoned for a long time for various reasons including:

- It was discovered that besides gravitation and electromagnetism there are the weak and strong nuclear forces of the Standard Model not accounted for by this scenario.

- The 5-dimensional action $S = \frac{1}{\kappa^2}\int \mathcal{L}\mathrm{d}^5x$ comes with an overall factor

[2]The interpretation of the scalar terms is a little more subtle and we will ignore them until we learn how to couple scalars to gravity.

[3]One way to see this is to remember that the components obey a wave equation. Therefore the field has a wavelength $\frac{1}{|\mathbf{k}|}$ where $\mathbf{k}$ is the wave vector. To fit into the small extra dimension, this wavelength must be very small. Then $|\mathbf{k}|$ must be large but that means that the derivative $\partial_5 h$ is very large.

of the gravitational coupling $\frac{1}{\kappa^2}$. Show that this means that the electromagnetic coupling is $g_* \propto \kappa$. This is all wrong since as we learned in the first week of class, the gravitational coupling is many (~ 20) orders of magnitude weaker than the electromagnetic coupling.

With the advent of (super)string theory as a candidate theory of "everything" (an arrogant term meaning gauge and gravitational forces and the matter on which they act) the KK paradigm was resurrected. This is necessarily so because flat Minkowski space is a solution (read valid background) for this theory only if the spacetime dimension is 10. Then the above problems must be circumvented.

The former problem is avoided since gravity is replaced with a supersymmetric version called supergravity. There are 5 basic types, 2 of which have large gauge groups which can accommodate the standard model. (This automatically implies grand unification.)

The second problem is easily avoided by assuming the extra dimensions are *warped*. This means as you move into the extra dimensions, the metric scales conformally by an exponentially large factor. For example in one extra dimension we could have $g(x^a, x^5) = \mathrm{e}^{-|x^5|/L} g(x^a)$ for some length scale L depending on the model. At $x^5 = 0$ the 4-dimensional metric is unscaled. For $|x^5| >> L$ on the other hand, the gravitational field is heavily suppressed by the exponential "warp factor" and one can show from this that the resulting gravitational strength will be as well. Therefore, if our 4-dimensional world is in this region, gravity will be weak. One still has to show that the resulting electromagnetic force is not scaled similarly.

Write down the Maxwell Lagrangian in a 4-dimensional background with metric $g(x)$. (Look it up if you do not know how to derive it.) Rescale $g(x) \mapsto \Omega^2(x) g(x)$. What happens to the Lagrangian? This is called *conformal invariance*.[4] Would the same thing be true if you replaced the Maxwell Lagrangian with the Yang-Mills Lagrangian?

[4] When people talk about a Conformal Field Theory (CFT) they are referring to a quantum field theory (QFT) which has this symmetry. A QFT is a classical field theory which has been "quantized" meaning that it has been modified (deformed) to be consistent with the principles of quantum mechanics to be discussed next semester.

3 Noether current

In this section we will show Noether's theorem as it applies to Lagrangian systems of the type we have been discussing in class: For every symmetry of the action there is a conserved current. We assume that the action is a local functional on the a space of fields $\{\varphi_A\}$ where A is a generalized index which indicates which field we are talking about and simultaneously stands for the position x in spacetime as well as any other indices the field φ may carry. For example, if φ_A is a gauge field then $A = (x, a)$ and $\varphi_{(x,a)} = A_a(x)$.

Then consider the action with Lagrangian $\mathfrak{L}$ which depends on the fields and their first derivatives

$$S\left[\{\varphi_A\}\right] \;=\; \int \mathfrak{L}\left[\{\varphi_A, \partial\varphi_A\}\right] \mathrm{d}^4x \;. \tag{4}$$

By varying the action show that

$$\frac{\partial \mathfrak{L}}{\partial \varphi_A} = \frac{\delta S}{\delta \varphi_A} + \partial_a\left(\frac{\partial \mathfrak{L}}{\partial(\partial_a \varphi_A)}\right) \;. \tag{5}$$

(Note the difference between functional derivatives and partial derivatives.) Plug this into the chain rule for $\delta\mathfrak{L}\left[\{\varphi_A, \partial\varphi_A\}\right]$ to show that

$$\delta\mathfrak{L} = \frac{\delta S}{\delta \varphi_A}\delta\varphi_A + \partial_a\left(\frac{\partial \mathfrak{L}}{\partial(\partial_a \varphi_A)}\delta\varphi_A\right) \tag{6}$$

Now suppose that S is invariant under the change $\varphi_A \to \varphi_A + \delta\varphi_A$ for some variations $\delta\varphi_A$. Argue that this implies that $\delta\mathfrak{L} = \partial_a K^a$ for some K^a. Conclude from this and the calculations above that the *Noether current*

$$j^a(x) \equiv \frac{\partial \mathfrak{L}}{\partial(\partial_a \varphi_A)}\delta\varphi_A(x) - K^a(x) \tag{7}$$

is conserved

$$\partial_a j^a(x) = 0 \tag{8}$$

modulo equations of motion.

As an example, take the complex scalar field $\psi(x)$ minimally coupled to a U(1) gauge field $A_a(x)$ we discussed in class. Reconstruct the action by gauging the global U(1) symmetry with parameter α and compute the equations of motion for the fields. Show that the Noether current for the U(1) symmetry is

$$j^a(x) \;=\; i\alpha\,\bar\psi\,\overset{\leftrightarrow}{\nabla}{}^a\psi\;. \tag{9}$$

It is conventional to drop the symmetry parameter α from this expression. After doing this, show that this current is conserved using the equations of motion for ψ.

4 Classical field theory

In this problem we will get an idea of what it means to solve a classical field theory perturbatively. For ease of exposition we start with the simplest non-trivial classical field theory called "ϕ^4 theory".[5] Let $\phi : \mathbb{R}^{3,1} \to \mathbb{R}$ be a real scalar field with dynamics determined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial^a\phi\,\partial_a\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{10}$$

with coupling $\lambda > 0$. (Why is this condition on the coupling necessary?) The first term is the free kinetic term which gives the d'Alembertian acting on the field. The second term is quadratic like the first term and is related to the mass of the field.[6]

Write down the equation of motion for ϕ. Note that it is a non-linear second-order partial differential equation. We can attempt to "solve" such equations by Green's method (Homework 3).

First solve the free equation $\lambda = 0$. (We did this in class for the electromagnetic field strength; this case is easier since scalar fields have no polarization.) Write down the dispersion relation. Given the previous footnote, comment on the physical interpretation for the wave vector and frequency.

Now suppose we can find the solution to the equation

$$\left(\Box_x - m^2\right) G(x-y) = \delta^{(4)}(x-y)\;. \tag{11}$$

[5]This is usually pronounced "phi-four theory".

[6] We will not be able to explain this statement until we have discussed quantum mechanics. In the quantum theory the momentum p_a is replaced by an operator $\hat p_a$ which is represented on fields as $\hat p_a = -i\hbar\partial_a$ for a new fundamental parameter $\hbar$ with units of action called *(reduced) Planck's constant*. Then the linear terms in the field equation $(\Box - m^2)\phi = 0$ become $-(\frac{1}{\hbar^2}\hat p^2 + m^2)\phi = 0$ which gives the mass-shell equation $p^2 = -m^2$ for the eigenvalues of the momentum operator in units in which $\hbar = 1$.

Then we know that we can find the "solution" to the full equation of motion by convolution:

$$\phi(x) = \int G(x-y)[\lambda\phi(y)]^3 \,\mathrm{d}^4y + \phi_{\mathrm{free}}(x) \tag{12}$$

where ϕ_{free} is the solution to the free field equation. Of course this is not an explicit solution at all because $\phi(x)$ appears on the right-hand-side. Nevertheless this form is useful to find the solution perturbatively in the coupling parameter λ.

Expand $\phi(x)$ in a formal power series in λ as

$$\phi(x) = \sum_{n=0}^{\infty} \lambda^n \phi_n(x) \tag{13}$$

and plug this into the formal solution (12). Separate this equation into equations with definite powers of λ. Write them out explicitly for λ^0 up to λ^3. Reduce the expression for ϕ_n to integral equations in terms of ϕ_{free} keeping track of coefficients. For example, for the $n=3$ case you should find

$$\begin{aligned}\phi_3(x) &= 9\int \mathrm{d}y \int \mathrm{d}y' \int \mathrm{d}y'' G(x-y)\phi_0(y)^2 G(y-y')\phi_0(y')^2 G(y'-y'')\phi_0(y'')^3 \\ &\quad +3\int \mathrm{d}y \int \mathrm{d}y' \int \mathrm{d}y'' G(x-y)\phi_0(y) G(y-y')\phi_0(y')^2 G(y'-y'')\phi_0(y'')^3\end{aligned} \tag{14}$$

(Hint: It helps to invent some shorthand for these calculations such as $\phi_3 = 9G\phi_0^2G\phi_0^2G\phi_0^3 + 3G\phi_0G\phi_0^3G\phi_0^3$ as long as you can keep track of at what point in spacetime everything is occurring. Please write your answers out fully though.)

This process is clearly a pain to carry out to higher orders. We therefore invent a diagrammatic technique to keep track of the calculations. Start by looking at your solution of ϕ_1 in terms of ϕ_0

$$\phi_1(x) = \int \mathrm{d}y\, G(x-y)\phi_0(y)^3 \ . \tag{15}$$

Represent this pictorially by 2 dots corresponding to the points x and y and label the dots accordingly. Connect the points with a double line representing $G(x-y)$. This is called an *internal line*. From the dot corresponding to y draw 3 more lines (non-double!) in the direction away from x and ending anywhere. The endpoints of these 3 lines represent the 3 factors of $\phi_0(y)$. Note that these 3 lines don't "go to a different point" since the 3 ϕ_0 are all at y but are just there to keep track of these factors. These lines are called *external lines*. Let us call this diagram the basic diagram.

Now do the same for $\phi_2(x)$. This time there will be 3 points and 2 double lines connecting them since there are 2 factors of Green's function. The resulting diagram has 5 external lines corresponding to the powers $\phi_0(y)^2\phi_0(y')^3$. We see that this diagram can be built from two copies of the basic diagram corresponding to the points y and y' by gluing the double line of the basic y' diagram over one of the external legs emanating from y. Note that there are 3 ways to choose a line from the y vertex. This corresponds to the factor of 3 in your formula for ϕ_1.

Repeat this exercise for ϕ_2. Explain the factors of 9 and 3. Explain the emerging pattern. Explain what to do for the diagram corresponding to ϕ_n. Explain how to write down the formula for $\phi_n(x)$ from its diagram.

The diagrams and rules you have constructed are called the *Feynman diagrams* and *Feynman rules* for ϕ^4 theory in position space.[7] With a bit of thought you can convince yourself that this method will work for a more general potential $V(\phi)$ and even for theories with different types of fields with polynomial interactions. Note that all your diagrams will be *tree graphs*, that is, they have no loops. The only (but major) difference in the quantum (QFT) version of this classical field theory is that the diagrams are allowed to have loops.

5 Higgs Mechanism

Consider a complex *Higgs field* H^α ($\alpha = 1, 2$) in the $\mathbf{2}_{\frac{1}{2}}$ representation of the electroweak gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Write down the action for this field minimally coupled to the electroweak gauge bosons W_a^i and B_a. Add to this the *Higgs potential* from homework 2

$$V = -\mu^2 \bar{H}_\alpha H^\alpha + \frac{\lambda}{2}(\bar{H}_\alpha H^\alpha)^2 \ . \tag{16}$$

Argue that this potential is invariant under the electroweak symmetry. Recall, however, that $\lambda > 0$ for the system to have a ground state and that for

[7] The Fourier transform of this whole story gives the diagrams and rules in momentum space.

$\mu^2 > 0$, $(H^\alpha) = 0$ is not a ground state. Therefore, for the system to minimize it's energy, the Higgs field must have at least one non-zero component. By SU(2) invariance, we can choose this component to have τ^3 eigenvalue $-\frac{1}{2}$

$$(H^\alpha_*) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{17}$$

What electric charge does this non-zero component have? Note that although this is a legal ground state, it does not respect the original SU(2)×U(1) symmetry. Write out the action with this value of the Higgs field plugged in. Due to $v \neq 0$ there are now non-vanishing coefficients for the terms quadratic in W and B. Show that there is a linear combination of W^3_a and B_a which has no quadratic term. Call this combination A_a. Call the orthogonal combination Z^0_a. Finally, define $W^\pm_a = \frac{1}{\sqrt{2}}(W^1_a \pm iW^2_a)$.

We therefore have the following situation: The minimally coupled Higgs field describes a system the dynamics of which respects an $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge invariance but the ground state of the system breaks it to the $\mathrm{U}(1)_Q \subset \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ subgroup fixing H_*. The generator of the unbroken symmetry is the electric charge $Q = Y + \tau^3$.

Whenever the Lagrangian dynamics has a gauge symmetry with group G but the ground state of the system is invariant only under a proper subgroup $H \subset G$, we say that *G is spontaneously broken to H* by the ground state.

We therefore have that the electroweak symmetry is spontaneously broken to the electromagnetic symmetry by the Higgs' ground state. In addition to breaking the symmetry spontaneously, this non-zero value of the field gives a mass to the $W^\pm$ and Z^0 gauge bosons.[8] That the photon remains massless is a consequence of the unbroken electromagnetic $\mathrm{U}(1)_Q$. This is the *Higgs mechanism* used in quantum field theory to give masses to gauge bosons without ruining the underlying gauge symmetry.[9]

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[8]See footnote 6 for a partial explanation of this statement.

[9]Here is an analogy to a much more intuitive system: Suppose you have a cylindrical flexible metal rod standing upright on a table and perpendicular to it. Suppose you apply a force pushing directly down the rod in such a way that the whole system has perfect axial symmetry $\mathrm{SO}(2) \cong \mathrm{U}(1)$ in the plane of the table. With sufficient force, the rod will "buckle" or bend with the bulge popping out to one side. This configuration has no symmetry. If it were indeed possible to arrange things so that the original configuration was perfectly symmetric then the bulge must pick a direction at random breaking $\mathrm{SO}(2) \to \{1\}$ spontanously. This is similar to the Higgs field picking a direction at random in $\mathbb{C}^2$.