

MAT560 Homework 4

Due Monday, November 26th

Abstract

This may not be the complete assignment. More reading and/or one question may be added on or before Monday, November 19th.

1 Reading

Read §3.5, §11.1, and §14.1 in Frankel [1].

2 Addition of velocities in special relativity

Suppose observer $\bar{\mathcal{O}}$ is traveling relative to an inertial observer \mathcal{O} with constant speed v in the $\pm x$ -direction. Suppose $\bar{\bar{\mathcal{O}}}$ is traveling relative to $\bar{\mathcal{O}}$ with speed w also in the $\pm x$ -direction. Show that $\bar{\bar{\mathcal{O}}}$ is traveling relative to \mathcal{O} with speed

$$v \oplus w = \frac{v + w}{1 + vw} . \quad (1)$$

Note that $v \oplus w$ is sub-luminal for sub-luminal v and w and show that $v \oplus w = 1$ iff $v = 1$ or $w = 1$ or both.

Introduce the *rapidity* $\eta \in (-\infty, +\infty)$ associated to the velocity v by

$$\eta = \tanh^{-1} v \quad (2)$$

Using the addition rule $(v, w) \mapsto v \oplus w$ derive the rule for the associated rapidities.

Write the Lorentz transformation rules $\bar{t} = \gamma(t - vx)$, *et cetera* derived in class for a boost in terms of the rapidity η instead of the velocity v . Show that the interval ds^2 is invariant under boosts due to a hyperbolic-trigonometric identity. In what sense is a boost a “rotation”?

3 Weakly curved spacetime and Newtonian gravity

As you keep hearing in class and elsewhere, the gravitational force experienced by particles arises due to the curvature of space-time. In this background the particles are simply following trajectories which are “as straight as possible” *id est* geodesics. In this sense the gravitational field is the space-time metric g and the particle trajectories $x : \Sigma \rightarrow (M, g)$ obey the geodesic equation derived in homework 2

$$\frac{du^a}{d\lambda} + \Gamma_{bc}^a u^b u^c = 0 \quad (3)$$

where

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \quad (4)$$

are the Christoffel symbols.

We want to recover Newton’s law of gravitation. However, this law is non-relativistic and is expected to be valid only in the limit of weak gravitational fields. The former approximation is easy to make if the particle is massive $m > 0$ and has small 3-velocity $\|\mathbf{v}\| \ll 1$. For the momentum this means $E = p^0 \gg p^i$.

Implement this non-relativistic approximation on the geodesic equation (3) to derive a simplified expression for $dE/d\tau$ and $d\mathbf{p}/d\tau$. (The only Christoffel symbols which should contribute are Γ_{00}^0 and Γ_{00}^i .)

With regard to the weak field approximation we note that no gravitational field just corresponds to the Minkowski space, that is, the metric $g_{ab}(x) \equiv \eta_{ab}$. A weak gravitational field should therefore be a small correction $g_{ab}(x) = \eta_{ab} + h_{ab}(x)$ where all components $|h_{ab}| \ll 1$. From the symmetries of the Newtonian theory we know that the metric must be of the form

$$ds^2 = -(1 + 2\phi)dt^2 + (1 + f)\delta_{ij}dx^i dx^j \quad (5)$$

for some functions $\phi = \phi(x, t)$ (the 2 is for later convenience) and $f = f(x, t)$ obeying $|\phi| \ll 1$ and $|f| \ll 1$ everywhere on $\mathbb{R}^{3,1}$.

Einstein's theory of general relativity restricts the metric to obey a complicated non-linear differential equation called the Einstein equation. Since we have not derived this equation we must take on faith that our general ansatz for the metric (5) will solve this equation provided that $f = -2\phi$.

To first order in ϕ compute the Christoffel symbols you need for your non-relativistic approximation above. Use this to recover Newton's law for the gravitational force on a massive particle, thereby showing that ϕ is the gravitational potential. Furthermore, show that the energy of the particle is conserved provided ϕ is independent of time.

4 Free massless spin-2 field

In this problem we will use the combined strength of the 3 basic principles studied so far (action, relativity, and gauge) to construct the linearized dynamics of the gravitational field. As in the problem above, expand the metric $g_{ab} = \eta_{ab} + h_{ab}$. The action principle implies that we seek a functional $S[h]$ such that the variation with respect to h gives a deterministic equation for h . That we study only the linearized theory means that this equation of motion is linear in h , that is, $S[h]$ is quadratic in h . The principle of relativity implies, as usual, that S is a Lorentz scalar. In order to use the gauge principle we need to know the linearized gauge transformation for h .

In the non-linear theory for the metric (Einstein's theory of gravitation), the gauge symmetry is diffeomorphism invariance, that is, invariance under smooth, invertible changes of coordinates $x^a \mapsto y^a(x)$. Write down the transformation law for the rank-2 symmetric tensor field $g_{ab}(x)$. Supposing the transformation is close to the identity, $y^a(x) = x^a + \xi^a(x)$ for some vector field $\xi^a(x)$,¹ deduce the transformation of h to linear order in ξ .

Taking this rule to be the gauge transformation of $h_{ab} \mapsto h'_{ab}$ we want to construct an invariant action $S[h] = S[h']$. There are (at least) 2 ways to do this. The first would be to try to construct a "field strength" for h_{ab} and use this in the action. As we will see below, this is a bit tricky. Instead we take the "brute force" approach and write

¹This vector field is the one generating the flow induced by the diffeomorphism.

down all the terms we could put in the Lagrangian density. Prove that the most general Lagrangian density is equivalent to

$$2\mathcal{L} = h^{ab}\square h_{ab} + a h\square h + b h^{ab}\partial_a\partial^c h_{bc} + c h\partial^a\partial^b h_{ab} \quad (6)$$

where $h = \eta^{ab}h_{ab}$ is the trace of the fluctuation and a, b, c are constants. Perform a gauge transformation on each term in the action with this Lagrangian organizing the answer into terms proportional to $\partial \cdot \xi$ and terms which cannot be written like this. (Don't forget that you can integrate by parts!) Fix a, b, c by first canceling the terms proportional to $\partial \cdot \xi$ and then the rest.

Define

$$G_{ab}(h(x)) = \frac{\delta S[h]}{\delta h^{ab}(x)} \quad (7)$$

and compute the equation of motion from your gauge invariant action. Compute $\partial^a G_{ab}$. Could you have anticipated this? (Hint: Show that the variation of the action gives

$$\delta S = \int \delta h^{ab} G_{ab} d^4x \quad (8)$$

and consider the special case when the variation δh_{ab} is a gauge transformation.)

References

- [1] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*