MAT560 Homework 3

Due Wednesday October 29^{th}

1 Reading

Read §15 and sections 1 and 2 in §19 in Frankel [1].

2 Green's solution to Poisson's equation

In class we studied some aspects of the Newtonian theory of gravitation and arrived at the Poisson equation for the gravitational scalar potential field

$$\nabla^2 \varphi(\mathbf{x}) = 4\pi G_N \, \rho(\mathbf{x}) \tag{1}$$

in the presence of a source ρ.

Introduce *Green's function* $G(\mathbf{x}, \mathbf{y})$ for the Laplacian ∇^2 as[1]

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \; . \tag{2}$$

Note that the Green function is only defined up to addition by a harmonic function but that ignoring the kernel of the Laplacian, G is the "inverse" of ∇^2. In particular, show that supposing we can find Green's function G then for any source ρ this gives an integral equation (solution) for the field φ (up to a harmonic function φ_0).

Solve equation (2) to get a coordinate representation of $G(\mathbf{x}, \mathbf{y})$ by first solving it away from $\mathbf{x} = \mathbf{y}$ and then by integrating both sides of the equation and using Stoke's formula.[2]

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[1] The subscript on the Laplacian indicates that the differentiation is with respect to $\mathbf{x}$ only (that is not $\mathbf{y}$).

[2] Hint: The answer should be in your notes.

In many practical situations, the mass density ρ has support only in a small region $U \subset \mathbb{R}^3$ and we want to compute the field at a point $\mathbf{x}$ far outside of this region. The answer you derived above for the potential becomes an integral over all $\mathbf{y} \in U$. For points $\mathbf{x}$ far away from U we have that $|\mathbf{x}| \gg |\mathbf{y}|$ allowing us to expand $|\mathbf{x}-\mathbf{y}|^{-1} = 1/\sqrt{x^2 - 2\mathbf{x}\cdot\mathbf{y} + y^2}$ in $1/|\mathbf{x}|$ as

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{x}\left\{1 - \frac{1}{2}\left[-2\frac{\mathbf{x}\cdot\mathbf{y}}{x^2} + \frac{y^2}{x^2}\right] + \dots\right\} \tag{3}$$

where $x \equiv |\mathbf{x}|$. Using this, write your potential in the form of a *multi-pole expansion*

$$\frac{1}{4\pi G_N}\varphi(\mathbf{x}) = \frac{1}{x}M + \frac{x^i}{x^3}p_i + \frac{1}{2}\frac{x^i x^j}{x^5}Q_{ij} + \dots \tag{4}$$

and show that

$$\begin{aligned} M &= \int_U \rho \mathrm{d}v \\ p_i &= \int_U y_i \rho \mathrm{d}v \\ Q_{ij} &= \int_U \left(3y_i y_j - \delta_{ij} y^2\right) \rho \mathrm{d}v \ . \end{aligned} \tag{5}$$

Notice that Q_{ij} is symmetric and traceless. That is, it is an irreducible representation of the spacial isometry group SO(3). In fact, the expansion above is a decomposition of a general mass distribution ρ into irreducible representations of SO(3).[3]

The first so-called *moment* of ρ is the *monopole moment* M corresponding to the trivial representation of SO(3) and gives the total mass of the distribution. The second moment p_i is called the *dipole moment* and corresponds to the defining representation of SO(3). The third moment Q_{ij} is the *quadrupole moment*. The 6-dimensional symmetric tensor representation of SO(3) is reducible because it contains a part proportional to the trace. As we have already accounted for this monopole moment, we should subtract it from this representation. The resulting 5-dimentional representation is irreducible and it is this one which corresponds to the quadrupole moment.

[3]This is analogous to a Fourier decomposition which is the decomposition of a holomorphic function on the complex plane into irreducible representations of U(1)≅SO(2).

3 Classical non-relativistic spinors

Write down a 3×3 matrix $R^i{}_j(\theta; z)$ which rotates a coordinate vector in $\mathbb{R}^3$

$$x^i \mapsto R^i{}_j(\theta; z) x^b \tag{6}$$

around the $+z$-axis[4] by an angle θ, preserving it's length. Do the same for the y- and x-axes. Show that these matrices form a basis for SO(3). Define the 3 constant matrices

$$L^i := \frac{\mathrm{d}}{\mathrm{d}\theta} R(\theta; i)\Big|_{\theta=0} \tag{7}$$

Compute them from your expressions above (are they skew-symmetric?) and show that they obey the commutation relations[5]

$$\left[L^i, L^j\right] = \sum_{k=1}^{3} \epsilon^{ijk} L^k \ . \tag{8}$$

These are the defining relations of the Lie algebra $\mathfrak{so}(3)$.

Define the exponential as a homomorphism

$$\exp : \mathfrak{g} \ \rightarrow \ G \tag{9}$$

from the Lie algebra $\mathfrak{g}$ to the Lie group G. Taking $A \in \mathfrak{g}$, $\exp(A)$ is defined by the power series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \ . \tag{10}$$

Show that taking $A = \theta L^z$ gives $\mathrm{e}^A = R(\theta; z)$, that is, L^z generates an infinitesimal rotation around the $+z$-axis.

Define the *Dirac matrices* $\{\gamma^i\}_{i=1}^3 \in M(3, \mathbb{C})$ to be a set of matrices satisfying the *Clifford algebra relations*

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij} \mathbf{1}_3 \tag{11}$$

where $\{A, B\} \equiv AB + BA$ is the anti-commutator of A and B. Use the Dirac matrices to construct a new set of matrices

$$\Sigma^i = -\frac{1}{4} \sum_{j,k=1}^{3} \epsilon^{ijk} \left[\gamma^j, \gamma^k\right] \ . \tag{12}$$

[4] Use the right-hand-rule $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = +\hat{\mathbf{z}}$.

[5] Normalize $\epsilon^{123} \equiv \epsilon^{xyz} = +1$.

Using the Clifford relations to commute the Dirac matrices through one another show that the new matrices satisfy the $\mathfrak{so}(3)$ relations[6]

$$\left[\Sigma^i, \Sigma^j\right] = \sum_{k=1}^{3} \epsilon^{ijk} \Sigma^k \ . \tag{13}$$

Show that when the matrices are 3×3,

$$(\Sigma^i)^j{}_k = -\sum_{m=1}^{3} \epsilon^{ijm} \delta_{mk} \tag{14}$$

does the trick.[7]

Now define the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{15}$$

Show that they obey the Clifford algebra relations

$$\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij} \mathbf{1}_2 \ . \tag{16}$$

This implies that the new matrices

$$\hat{\Sigma}^i = -\frac{1}{4} \sum_{j,k=1}^{3} \epsilon^{ijk} \left[\sigma^j, \sigma^k\right] \tag{17}$$

satisfy the same algebra as the Σs. (Note that they are skew-hermitian as were the L^is.) In particular $\hat{\Sigma}^i$ performs an infinitesimal transformation in $\mathbb{C}^2$ corresponding in $\mathbb{R}^3$ to a rotation around the i^{th} axis.

In this very special case, $\hat{\Sigma}^i = -i\sigma^i$. Show that ($-i$ times) the Pauli matrices form a basis for the Lie algebra $\mathfrak{su}(2)$. We have found therefore that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as Lie algebras.

In general SO(n) is not simply connected. The universal cover of this group is called Spin(n). Although we have not quite proven it, it is not hard to show now that Spin(3) $\cong$ SU(2) having the topology of a 3-sphere is the double cover of SO(3) which has the topology $\mathbb{R}P^3$.

[6]Hint: Start with $\gamma^m \gamma^n \Sigma^j$ and use the Clifford algebra to push the γ-matrices to the right until you get to $\Sigma^j \gamma^m \gamma^n +$ a bunch of terms. Then contract both sides with ϵ^{imn} and reconstruct Σ^i.

[7]Hint: Write a few of them out and compare to your calculations involving L^i.

Now, the γ-matrices, being 3×3 matrices naturally act on the space $\mathbb{R}^3$ of 3-dimensional vectors. However, we have just seen that the algebra of the 3-dimensional rotation group admits a (complex) 2-dimensional representation which acts naturally on $\mathbb{C}^2$. This space is called the space of *spinors*.

Similarly to the fact that the coordinates of a particle $x : \Sigma \to \mathbb{R}^3$ transform as vectors under rotations SO(3), we can consider a new type of "particle" $\theta : \Sigma \to \mathbb{C}^2$ transforming in the fundamental representation of Spin(3).

While products of coordinates of vectors commute

$$x^a x^b = x^b x^a \ , \tag{18}$$

for reasons which will not become apparent until much later (the spin-statistics theorem of relativistic quantum field theory), the components of a spinor should be taken to anti-commute

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha \tag{19}$$

Assuming this, the pair (x, θ) represents a map from Σ to the *superspace* $\mathbb{R}^3 \times \Pi\mathbb{C}^2$ which was discussed in the RTG seminar. The spinor θ is called the super-partner of x and the pair of maps is said to represent a *super-particle*. We will return to this topic further along in the course.

References

[1] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*