MAT560 Homework 2

Due Wednesday October 3^{rd}

1 Reading

Chapters 9 and 10 in Frankel [1].

2 Particle in a gravitational background

Consider a particle of mass m propagating on a Riemannian manifold M with metric g. Define the *Christoffel symbol*[1]

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) \ . \tag{1}$$

Construct the action $S[x(t)]$ for the particle making sure it has the right units. Compute the momentum conjugate to x^a. Is it what you expect? Compute the Euler-Lagrange equation and write it in a way that uses the Christoffel symbols. (It should look nice if done correctly.) What is this equation called? If you don't know, what would you call it?

As we will start to talk about in the coming weeks, in Einstein's theory of General Relativity the gravitational force arises from the curvature of the space(-time). The metric g above is called a gravitational background because the geometry of M sources the acceleration of the particle.

[1] This defines the Levi-Civita connection ∇ (torsion-free and compatible with the metric $\nabla_k g_{ij} = 0$) such that for any vector v^i the covariant derivative is given by $\nabla_i v^k = \partial_i v^k + \Gamma^k_{ij} v^j$, for any 1-form ω_i, $\nabla_i \omega_j = \partial_i \omega_j - \Gamma^k_{ij} \omega_k$, *et cetera*.

3 Particle in a magnetic background

The potential energy V of a particle was introduced in class by putting a (trivial) real line bundle on M. Suppose instead that we equip M with a 1-form qA where $q \in \mathbb{R}$ is the *coupling* of the particle to A. Assuming M is flat and 3-dimensional, construct the natural action for the particle propagating on this background. Define the *field strength* of A by

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{2}$$

and compute the Euler-Lagrange equation.

The magnetic field in physics courses is often introduced as a vector $\mathbf{B}$. Assuming

$$B^i = \frac{1}{2}\epsilon^{ijk} F_{ij} \tag{3}$$

show that the equation of motion reads

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{4}$$

where $\mathbf{v}$ is the velocity of the particle. Sketch the trajectory of a free particle with $q > 0$ with non-zero velocity entering a region of magnetic field. What about $q < 0$. Derive an expression for the radius of the circle traced out by a charged particle trapped in a magnetic field.

Compute the momentum conjugate to x^a. Is it what you expected? This is an example of why the momentum conjugate to a coordinate is often referred to as a "generalized" momentum.

Suppose A is exact. What happens to the magnetic field? What happens to the action? We see that we can change

$$A \mapsto A + \mathrm{d}\lambda \tag{5}$$

for any function $\lambda : M \to \mathbb{R}$ *without affecting the dynamics.* This inherent ambiguity in the definition of A is called a *gauge ambiguity* and the transformation (5) is called a *gauge transformation*. Note that as claimed in class, since the action is invariant under the gauge transformation, the equation of motion is co-variant – indeed invariant – under the transformation.

As we will discuss at length further along in the course, the transformation (5) is the transformation of a connection on a principle U(1) bundle over M and the field strength F_{ij} is its curvature. It turns out that this picture generalizes to different groups including non-abelian ones.[2] Furthermore all fundamental forces of Nature are gauge theories in the sense described by this problem!

4 How not to write the Lagrangian

In class we went through the following series of steps to obtain the Euler-Lagrange equations:

1. The equation of motion is $\dot{p}_a = F_a$
2. The momentum conjugate to x^a is defined by $p_a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}^a}\right)$.
3. The generalized force is $F_a = -\frac{\partial V}{\partial x^a}$.
4. The equation of motion can therefore be written as $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a}$ with $L = T - V$.

Of course we could also have switched 2 signs and tried

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = -\frac{\partial L}{\partial x^a} \tag{6}$$

with $L = T + V$ instead. Why would this have been wrong? (Hint: Can you construct an action which will give this new equation (6) as an equation of motion?)

5 Higgs' potential

Define the radial coordinate $r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ in $\mathbb{R}^3$ and consider the potential energy function

$$V(r) = \mu r^2 + \frac{\lambda}{4} r^4 \ . \tag{7}$$

What are the units of μ and λ? Construct from these parameters a quantity with the units of length.

[2] QCD, the theory Dan Freed mentioned in our field trip, is obtained by replacing U(1)→SU(N_c) where N_c is the number of different charges called "colors".

By insisting that the dynamics of a particle in this potential is well-defined, that is the potential is *stable*, argue that λ cannot be negative. Continuing this line of reasoning, discuss the various qualitatively different kinds of stable potential we can have for various points in the parameter space (μ, λ). What are the ground states for these different potentials? Show in particular that in some region of parameter space there is an entire 2-sphere of degenerate ground states. Sketch a graph of V for a 2-dimensional version of this potential $V(x, y)$. What is the radial distance r_* to this ground state in terms of μ and λ? What is the curvature of the graph $V(r)$ at $r = r_*$?

In the situation above, the dynamics of the theory defined by the potential (7) is invariant under rotations regardless of the point in parameter space. For some choices of parameters, the ground state is $r = 0$ respecting this symmetry. For other choices however, $r = r_* \neq 0$ and any choice of ground state $\mathbf{x}_*$ with $\mathbf{x}_* \cdot \mathbf{x}_* = r_*^2$ breaks the rotational symmetry SO(3)$\rightarrow$ U(1). In the particle physics literature this trivial-seeming effect in which the ground state does not respect the symmetry of the dynamics is called *spontaneous symmetry breaking*. The Higgs mechanism uses this effect to give mass to the particles in the standard model. The particle which breaks the symmetry is called the Higgs boson.

References

[1] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*