

Algebra II: Solutions to problems 4, 5 of the final exam

4. Denote by R the ring of all polynomials in the coefficients a_{ij} of an $n \times n$ matrix A . Show that polynomials in R invariant under the transformation $A \rightarrow CAC^{-1}$ for any $C \in GL_n(\mathbb{C})$ form a subring of R . Describe this subring explicitly in terms of generators and relations.

Solution: Denote this subring by S . If $f \in S$, then $f(A)$ is equal to the value of f on the normal Jordan form of A . Hence, when the normal Jordan form of A is diagonal, we have that $f(A)$ depends only on the eigenvalues of A . Define the polynomial $P_f(x_1, \dots, x_n)$ in n variables as follows: $P_f(\lambda_1, \dots, \lambda_n)$ equals to the value of f on a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$. Then for any matrix A (not necessarily diagonalizable) we have that $f(A) = P_f(\lambda_1(A), \dots, \lambda_n(A))$, where $\lambda_1(A), \dots, \lambda_n(A)$ are the eigenvalues of A . Indeed, this identity holds for all diagonalizable matrices, and those are dense in the set of all matrices. It is now easy to check that the map $f \rightarrow P_f$ provides an isomorphism between S and the ring of symmetric polynomials in n variables.

5. Consider the tautological representation of S_n on \mathbb{C}^n (i.e. S_n acts by permutations of basis vectors).

(a) Show that the tautological representation is the sum of the trivial one-dimensional representation and $(n-1)$ -dimensional irreducible representation (called *simplicial*), i.e. $\mathbb{C}^n = V^1 \oplus V^{n-1}$ where V^1 and V^{n-1} are irreducible S_n -invariant subspaces. Describe the action of S_n on V^{n-1} .

(b) Let χ be the character of the exterior square of the tautological representation. Find the inner product (χ, χ) .

(c) Using part (a) show that the exterior square of the simplicial representation is irreducible.

Hint: decompose the exterior square of the tautological representation using the decomposition $\mathbb{C}^n = V^1 \oplus V^{n-1}$. Use that a representation is irreducible iff the square (χ, χ) of its character χ is 1.

Solution: (a) Denote by e_1, \dots, e_n the basis vectors in \mathbb{C}^n . The vector $v = e_1 + \dots + e_n$ is fixed under any permutation from S_n . Hence, $V^1 = \langle v \rangle$ is invariant. The $(n-1)$ -dimensional subspace

$$V^{n-1} = \{x_1 e_1 + \dots + x_n e_n : \sum_{i=1}^n x_i = 0\}$$

is also invariant since the action of S_n preserves the sum of coordinates of any vector. Let us show that V^{n-1} is irreducible. Take any nonzero vector $u = x_1 e_1 + \dots + x_n e_n \in V$. Since the sum of coordinates is zero, at least two of them are distinct. Assume that $x_1 \neq x_2$. Then $u - (x_2/x_1)u = (x_1 - x_2)(e_1 - e_2)$, and V^{n-1} contains $e_1 - e_2$. Then it contains the vectors $e_1 - e_i$ for $i = 2, \dots, n$, which form the basis in V^{n-1} . Hence, the minimal S_n -invariant subspace containing u is V^{n-1} .

The group S_n acts on V^{n-1} by the isometries preserving the regular simplex with the vertices $e_1 - \frac{1}{n}v, \dots, e_n - \frac{1}{n}v$.

(b) First, let us find $\chi(\sigma)$ for each $\sigma \in S_n$. The diagonal element of the matrix of $\Lambda^2\sigma$ in the basis $\{e_i \wedge e_j : 1 \leq i < j \leq n\}$ corresponding to the basis vector $e_i \wedge e_j$ is 0 if σ does not preserve the set $I = \{i, j\}$, 1 if $\sigma(i) = i$ and $\sigma(j) = j$, and -1 if $\sigma(i) = j$ and $\sigma(j) = i$. Hence,

$$\chi(\sigma) = \sum_{I \subset \{1, \dots, n\}, |I|=2} \sigma_I,$$

where

$$\sigma_I = \begin{cases} 0, & \text{if } \sigma(I) \neq I, \\ \text{sgn}(\sigma|_I), & \text{otherwise.} \end{cases}$$

We now compute (χ, χ) :

$$n!(\chi, \chi) = \sum_{\sigma \in S_n} \left(\sum_{I, |I|=2} \sigma_I \cdot \sum_{J, |J|=2} \sigma_J \right) = \sum_{\sigma \in S_n} \sum_{I, J} \sigma_I \sigma_J.$$

Change the order of summation

$$n!(\chi, \chi) = \sum_{I, J} \sum_{\sigma \in S_n} \sigma_I \sigma_J$$

and evaluate

$$\sum_{\sigma \in S_n} \sigma_I \sigma_J.$$

for every pair of subsets I and J . It is easy to show that we get 0 if I and J are disjoint, $2(n-2)!$ if $I = J$, and $(n-3)!$ if $|I \cap J| = 1$. E.g. if $I = J$, then the number of permutations preserving I is $2(n-2)!$ and for all of them $\sigma_I^2 = (\pm 1)^2 = 1$. Hence,

$$\sum_{I, J} \sum_{\sigma \in S_n} \sigma_I \sigma_J = 2(n-2)! \sum_I 1 + (n-3)! \sum_{|I \cap J|=1} 1 = n! + n!,$$

and $(\chi, \chi) = 2$.

(c) Since $\mathbb{C}^n = V^{n-1} \oplus V^1$, we have that $\Lambda^2 \mathbb{C}^n = \Lambda^2 V^{n-1} \oplus V^{n-1}$. Then

$$2 = (\chi, \chi) = (\chi(\Lambda^2 V^{n-1}), \chi(\Lambda^2 V^{n-1})) + (\chi(V^{n-1}), \chi(V^{n-1})) + (\chi(\Lambda^2 V^{n-1}), \chi(V^{n-1})). \quad (1)$$

Note that for any two representations V and W of a finite group, we have the following inequalities

$$(\chi(V), \chi(V)) \geq 1; \quad (\chi(V), \chi(W)) \geq 0.$$

Indeed, if $V = V_1 \oplus \dots \oplus V_n$ and $W = W_1 \oplus \dots \oplus W_m$ are the decompositions into irreducible representations, then

$$(\chi(V), \chi(W)) = \sum_{i,j=1}^{n,m} (\chi(V_i), \chi(W_j))$$

and each term in this sum is either 1 or 0, hence nonnegative (in other words, $(\chi(V), \chi(W)) = \dim \text{Hom}_G(V, W)$). These inequalities applied to $\Lambda^2 V^{n-1}$ and V^{n-1} together with equation (1) imply that $(\chi(\Lambda^2 V^{n-1}), \chi(\Lambda^2 V^{n-1})) = 1$.