Algebra II: Midterm Test

due on March 14

1. "Jordan canonical form" Let A be a 2×2 matrix with complex entries:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define a linear operator $\mathcal{A}$ on the space of 2×2 matrices with complex entries by the formula

$$\mathcal{A}(X) = AX - XA.$$

Find the Jordan canonical form of the operator $\mathcal{A}$.

2. "Signature" Find the signature of the quadratic form $A \mapsto \mathrm{Tr}(A^2)$ on the space of $n \times n$ matrices with real coefficients.

3. "Tensor products" Let $R = \mathbb{C}[x]$ be the polynomial ring. Define R-modules M and N as follows. The module M is the vector space $\mathbb{C}^6$, where x acts as the operator

$$A = \begin{pmatrix} 0 & 0 & 30 & 0 & 0 & 0 \\ 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

The module N is the vector space $\mathbb{C}^5$, where x acts as the opertor

$$A = \begin{pmatrix} 0 & 0 & 0 & -5 & 0 \\ 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & -8 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

Find the tensor product $M \otimes_R N$. If we regard $M \otimes_R N$ as a complex vector space, what is its dimension? How does x act on this vector space?

4. "Skew planes" Let $\mathbb{R}^5$ be the Euclidean space with the standard inner product, and let P_1 and P_2 be the planes in $\mathbb{R}^5$ given by the equations

$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_3 + x_4 - x_5 = 0 \end{cases} \text{and} \begin{cases} x_1 - x_2 - 2x_3 = 0 \\ 2x_2 + x_3 + x_5 = 1 \\ x_3 + x_4 - 2x_5 = -2 \end{cases},$$

respectively. Find the distance between P_1 and P_2. (By definition, the distance between the planes P_1 and P_2 is the minimal possible distance between a point in P_1 and a point in P_2.)

5. "Hyperbolic motions" Let $PSL_2(\mathbb{R})$ be the quotient of $SL_2(\mathbb{R})$ by its center. Show that the groups $PSL_2(\mathbb{R})$ and $SO_{(2,1)}(\mathbb{R})$ are isomorphic.

6. "Euclidean motions" Let $SU_2(\mathbb{C})$ be the group of all unitary matrices with determinant one, and $PSU_2(\mathbb{C})$ its quotient by the center. Show that $PSU_2(\mathbb{C})$ is isomorphic to $SO_3(\mathbb{R})$.

7. "Symmetric and exterior powers" Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a nondegenerate linear operator. Denote by S^iA and Λ^iA the i-th symmetric and exterior powers of A, respectively. Prove the following identity in the ring of formal power series $\mathbb{C}[[z]]$:

$$\left(\sum_{i=0}^{\infty} \mathrm{Tr}(S^iA)z^i\right) \cdot \left(\sum_{i=0}^{\infty} (-1)^i \mathrm{Tr}(\Lambda^iA)z^i\right) = 1$$

8. "Circulant" Let C be the operator on $\mathbb{C}^n$, whose matrix in the standard basis is

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & \cdots & c_n & c_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{n-3} & c_{n-2} \\ c_n & c_1 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}.$$

Is C normal with respect to the standard Hermitian product on $\mathbb{C}^n$?