Algebra II: Homework assignment 9
Due date: April 28

1. Let f be a polynomial of degree n over a field F, and let D be its discriminant, that is

$$D = \prod_{i<j}(x_i - x_j)^2,$$

where x_1, x_2, ..., x_n are roots of f. Assume that $char(F) = 0$. Show that the Galois group of f is contained in the alternating subgroup A_n if and only $\sqrt{D}$ lies in F.

2. Let $K \supset L \supset F$ be fields. Prove or disprove:
(a) If K/F is Galois, then K/L is Galois.
(b) If K/F is Galois, then L/F is Galois.
(c) If K/L and L/F are Galois, then K/F is Galois.

3. Let $f(x)$ be a monic polynomial of degree n with roots $x_1,\ldots, x_n$. Let s_i be the elementary symmetric function of degree i in the roots and define $s_i = 0$ for $i > n$. Let $p_i = x_1^i + \ldots + x_n^i$ be the sum of the i-th powers of the roots of $f(x)$.
(a) Prove *Newton's formulas*:

$$p_1 - s_1 = 0; \quad p_2 - s_1p_1 + 2s_2 = 0; \quad p_3 - s_1p_2 + s_2p_1 - 3s_3 = 0; \ldots;$$

$$p_i - s_1p_{i-1} + s_2p_{i-2} - \ldots + (-1)^{i-1}s_{i-1}p_1 + (-1)^i i s_i = 0.$$

(b) Do $p_1,\ldots, p_n$ generate the ring of symmetric polynomials in $x_1,\ldots, x_n$?

4. Let f be an irreducible polynomial with rational coefficients.
(a) Show that if one of the roots of f is constructible by compass and straightedge, then the other roots are also constructible.
(b) Suppose that the degree of f is 4. Suppose that a root of f is constructible by compass and straightedge. What can you say about the Galois group of f? List all possibilities.
(c) Prove that the roots of f are constructible if and only if the degree of the splitting field of f is a power of 2.

5. Show that the polynomial $f(x) = x^4 + px + p$ is irreducible for every prime p and for $p \neq 3$, 5 has Galois group S_4. Prove that the Galois group for $p = 3$ is dihedral of order 8 and for $p = 5$ is cyclic of order 4.