Algebra II: Homework assignment 6
Due date: March 31

1. Find the degrees of the following field extensions over $\mathbb{Q}$:
(a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$,
(b) $\mathbb{Q}(e^{2\pi i/k})$ for $k = 8$, 9 and 10,
(c) a splitting field of the polynomial $x^3 - 3x + 1$
(d) a splitting field of the polynomial $x^3 + 3x + 1$

2. Trisect the angle 27° using compass and straightedge.

3. Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

4. Let F be any field. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

5. Let $\Phi_n(x)$ be the n-the cyclotomic polynomial.
(a) Prove that

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

(b) Prove that if p is prime, then $\Phi_p(x)$, regarded as a polynomial in $\mathbb{F}_p[x]$, is equal to $(x-1)^{p-1}$.

6* Bonus. Let $p_1, \ldots, p_k$ be mutually distinct prime numbers. Find the degree of the extension $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_k})$ over $\mathbb{Q}$.