

Algebra II: Homework assignment 5

Due date: March 24

1. Let R be a ring with 1, and let P_1 and P_2 be R -modules. Prove that $P_1 \oplus P_2$ is a projective R -module if and only if both P_1 and P_2 are projective.

2. Let A be a nonzero finite abelian group. Prove that A is not an injective \mathbb{Z} -module.

3. Let R be a P.I.D. Show that any finitely generated R -module M has a projective resolution of length 2. I.e. there is a short exact sequence

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

such that P_0 and P_1 are projective.

4. Let $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ be a short exact sequence of cochain complexes. Prove that if any two of A^\bullet , B^\bullet , C^\bullet are exact, then so is the third.

5. Prove that a finitely generated abelian group A is free if and only if $\text{Ext}^1(A, \mathbb{Z}) = 0$.

6. Let A be an arbitrary abelian group, and $C = \mathbb{Z}/m\mathbb{Z}$. Prove that there is a bijection between $\text{Ext}^1(C, A)$ and the set of equivalence classes of extensions of C by A .

Hint: Show that every such extension can be defined as the Cartesian product $A \times C = \{(a, k) : a \in A, k = 0, 1, \dots, m-1\}$ with addition given by the formula

$$(a_1, k_1) + (a_2, k_2) = \begin{cases} (a_1 + a_2, k_1 + k_2), & \text{if } k_1 + k_2 < m \\ (a_1 + a_2 + a, k_1 + k_2 - m), & \text{if } k_1 + k_2 \geq m \end{cases}$$

for some $a \in A$. Then find out when two extensions corresponding to different a are equivalent.

Or equivalently, every extension is the quotient of the group $A \oplus \mathbb{Z}$ by an infinite cyclic subgroup generated by $(a, -m)$ for some $a \in A$.