Algebra II: Homework assignment 4
Due date: March 2

1. (a) Show that the group $SO_2(\mathbb{R})$ is isomorphic to the group $SU_1(\mathbb{C}) = \{z \in \mathbb{C} : |z| = 1\}$
(b) Show that $SO_{(1,1)}(\mathbb{R})$ is isomorphic to $\mathbb{R}^*$.

2. Let A be an operator on the Euclidean vector space $\mathbb{R}^n$, and let $U \subset \mathbb{R}^n$ be an invariant subspace of A. Show that the orthogonal complement $U^\perp$ is an invariant subspace of the adjoint operator A^*.

3. Let V be a finite-dimensional vector space over an algebraically closed field. Prove that any collection (possibly infinite) of pairwise commuting linear operators on V has a common eigenvector.

4. (a) Let V be a vector space of dimension m over a field $\mathbb{F}$. Suppose that there exists a *nondegenerate* skew-symmetric bilinear form on V. Show that then m is even and there is a basis in V such that the matrix of the form with respect to this basis is

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & 0 & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & 0 & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Pfaffian:
(b) Solve problem 3(c) (about Pfaffian) from Homework 3, if you have not already done so.
Hint: Let $F = \mathbb{Q}(a_{ij})$ be the field of rational functions in variables a_{ij} with $1 \le i < j \le 2n$. Consider the skew-symmetric matrix (a_{ij}), $a_{ij} = -a_{ji}$ as the matrix of a skew-symmetric form on the coordinate vector space of dimension $2n$ over $\mathbb{F}$. Then apply part (a).

5. Let V be a vector space of dimension n over reals, and $A : V \to V$ a linear operator. Define the complexified operator $A_{\mathbb{C}}$ on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ by the formula

$$A_{\mathbb{C}}(v \otimes z) = A(v) \otimes z.$$

Suppose that $A_{\mathbb{C}}$ has an eigenbasis in $V_{\mathbb{C}}$. Show that there exists a basis in V such that the

matrix of A in this basis is

$$\begin{pmatrix} a_1 & b_1 & & & & & & & \\ -b_1 & a_1 & & & & & & 0 & \\ & & a_2 & b_2 & & & & & \\ & & -b_2 & a_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & a_k & b_k & & \\ & & & & & -b_k & a_k & & \\ & & & & & & & c_1 & \\ & & 0 & & & & & & \ddots \\ & & & & & & & & & c_l \end{pmatrix}$$

for some real numbers $a_1, \ldots, a_k$, $b_1, \ldots b_k$ and $c_1, \ldots, c_l$ (of course, $2k + l = n$).