Algebra II: Homework assignment 3
Due date: February 23

Symmetric and alternating forms.

1. Let ω be an n-linear form on a vector space V over a field $\mathbb{F}$. Assume that $\mathrm{char}(\mathbb{F}) \neq 2$. Show that ω is alternating, that is

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = sgn(\sigma)\omega(v_1, \ldots, v_n)$$

for any permutation $\sigma \in S_n$ and any vectors $v_1, \ldots, v_n \in V$, if and only if

$$\omega(v_1, \ldots, v_n) = 0$$

whenever $v_i = v_j$ for some $i \neq j$.

2. (a) Let q be a quadratic form on a vector space V. Show that q satisfies the parallelogram law:

$$2(q(u) + q(v)) = q(u + v) + q(u - v)$$

for any $u, v \in V$. (For any parallelogram, the sum of squares of its sides is equal to the sum of squares of its diagonals.)

(b) Let q be a homogeneous function of degree 2 on V . Show that q satisfies the parallelogram law if and only if q is polynomial.

3. (a) Show that any alternating bilinear form on an odd-dimensional vector space is degenerate.

Pfaffian:

(b) Prove that the determinant of a skew-symmetric 4×4 matrix

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

is the square of some polynomial in a, b, c, d, e, f.

(Bonus c*) Prove part (b) for a $2n \times 2n$ skew-symmetric matrix.

Euclidean spaces.

Let V be a finite-dimensional Euclidean space.

4. (a) Prove that every self-adjoint operator on V has an eigenvector.

(b) Prove that every orthogonal operator on V has a two-dimensional invariant subspace and the restriction of the operator to this subspace is a rotation around the origin.

5. Let E be an ellipsoid (in a 3-dimensional Euclidean space) with the semi-axes of lengths a, b and c. Find the length of a spatial diagonal of a rectangular parallelepiped whose faces are all tangent to E. In particular, show that this length does not depend on the choice of a parallelepiped.

Bonus 6* (Isomorphisms of classical groups). Denote by $SL_n(\mathbb{C})$ and $SO_n(\mathbb{C})$ the groups of all linear operators on $\mathbb{C}^n$ that preserve a non-degenerate alternating n-form and a non-degenerate symmetric bilinear form on $\mathbb{C}^n$, respectively.

Construct explicit isomorphisms:

(a)

$$SL_2(\mathbb{C})/\{\pm I\} \simeq SO_3(\mathbb{C}),$$

(b)

$$(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/\{\pm(I, I)\} \simeq SO_4,$$

(c)

$$SL_4(\mathbb{C})/\{\pm I\} \simeq SO_6(\mathbb{C}).$$

Hint: use tensor, symmetric etc. products to produce, say in part (a), a 3-dimensional vector space with a symmetric bilinear form from a 2-dimensional vector space with an alternating bilinear form.