Algebra II: Homework assignment 1
Due date: February 9

Tensor product of modules.

1. Find the tensor product of the following $\mathbb{Z}$-modules:
(a) $\mathbb{Z}^k \otimes_{\mathbb{Z}} \mathbb{Q}$
(b) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$

2. Let V be a complex vector space of dimension n. Denote by $V_{\mathbb{R}}$ the same space regarded as a vector space over reals.
(a) Find $\dim V_R$.
(b) Find $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$
(c) Find $V \otimes_{\mathbb{C}} \mathbb{C}$
Are complex vector spaces you get in parts (b) and (c) isomorphic? Is any of them isomorphic to V?

Modules over Noetherian rings.

3. Let M be a finitely generated module over a Noetherian ring. Prove that any submodule of M is also finitely generated.

Hint: First, prove the statement when M is a free module. Then use that an arbitrary M can be represented as a quotient of a free module.

Modules over Principal Ideal Domains.

4. Let $R = \mathbb{C}[x]$ be the ring of polynomials with complex coefficients. Which of the following R-modules are isomorphic?
(a) The vector space $\mathbb{C}^2$ where x acts as the following operator

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(b) $(R \oplus R)/K$, where K is a submodule generated by $(x^3 - x^2 + 1, x(x+1))$ and $(x(x-1), x)$,
(c) $R/(x)$

5. Let R be as in Problem 4. Then a finitely generated R-module M of zero free rank is the same as a finite–dimensional complex vector space V with a linear operator $A : V \to V$ (=action of $x \in R$). Under what conditions on the operator A is the module M cyclic?

6. Let $R = \mathbb{C}[[x]]$ be the ring of formal power series with coefficients in $\mathbb{C}$.
(a) Classify all finitely generated R-modules.
(b) Let M be a finitely generated R-module of zero free rank. Show that M is a finite–dimensional complex vector space, and $x \in R$ acts on this space as a linear operator A. Under what conditions on A is the converse true (i.e. when a finite–dimensional complex vector space with a linear operator A can be endowed with a structure of an R-module so that $x \in R$ acts as A)?