Algebra I: Homework assignment 9
Due date: November 17

1. Classify up to similarity all $n \times n$ complex matrices A such that $A^n = I$.

2. **(a)** Find all eigenvectors of the operator A on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}.$$

What is its Jordan form? Give an example of a basis in $\mathbb{C}^2$ such that the matrix of A in this basis is the Jordan form of A.

(b) Let A be a 2×2 matrix with complex entries:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define the linear operator $\mathcal{A}$ on the space of 2×2 matrices with complex entries by the formula

$$\mathcal{A}(X) = AX$$

For each matrix A, find the Jordan canonical form of the operator $\mathcal{A}$.

3. Let A be an operator on a finite-dimensional vector space V. Define the exponent of A as the operator e^A obtained by taking the value of the exponential power series at A, i.e.

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots.$$

(a) Let P be an invertible operator on V. Prove that $Pe^AP^{-1} = e^{PAP^{-1}}$

(b) Prove that if A and B commute, then

$$e^{A+B} = e^A e^B.$$

(c) Compute the exponent of the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

(d) Prove that if A is antisymmetric (i.e. $A + A^t = 0$), then e^A is orthogonal (i.e. $AA^t = I$).

4. Let a_n be the sequence of numbers defined by the recurrent relation:

$$a_{n+2} = 2a_n - a_{n+1}, \quad a_0 = 1, a_1 = 1.$$

Find an explicit formula for a_n.

5. (a) Find a solution of the matrix equation:

$$X^2 = A, \text{ where } A = \begin{pmatrix} 3 & 7 \\ 6 & 2 \end{pmatrix}.$$

(b) Prove that every invertible $n \times n$ matrix has a square root. Is this statement true for non-invertible matrices?

Hint: show that if A has square root, then any similar matrix PAP^{-1} also has square root. Then take the Jordan form of A and compute separately square roots for all of its Jordan elementary matrices. Note that a square root of a matrix $I + N$, where N is a nilpotent matrix (i.e. $N^k = 0$ for some k), is the value of the binomial series for $(1+t)^{\frac{1}{2}}$ at $t = N$ (show that this series converges for $t = N$).

6. Let π be the reduction homomorphism

$$\pi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, \quad \pi : (s,t) \mapsto (s \pmod n), t \pmod n)).$$

Find all n such that $\pi(5,3)$ and $\pi(9,1)$ do not generate the group $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.