Algebra I: Homework assignment 8
Due date: November 10

1. Let U and V be subspaces of the same vector space. Verify that $U \cap V$ and $U + V = \{u + v : u \in U, v \in V\}$ are also subspaces. Prove that

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

2. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

3. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (*Hint:* Let $T = D$ be the differentiation operator on the space of polynomials.)

4. Let V be the vector space over of all real polynomial functions p of degree at most 2. For any fixed $a \in \mathbb{R}$ consider the *shift operator* $T : V \to V$ with $(Tp)(x) = p(x + a)$.
Explain why T is linear and find the image and the kernel of T. Is T an isomorphism? Write down the matrix of T with respect to the ordered basis $\mathcal{B} = \{1, x, x^2\}$.

5. **(a)** Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.

(b) Let $T, U \in \text{End}(V, V)$ be linear operators on the finite dimensional vector space V. Prove that the rank of the composition UT is less than or equal to the minimum of the ranks of T and U.

6. Find the number of k-dimensional subspaces in an n-dimensional vector space over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.