Algebra I: Homework assignment 7
Due date: October 27

1. Prove that all groups of order 200 are not simple.

2. Prove Cauchy theorem for all finite groups.
Cauchy's theorem: If G is a finite group and p is a prime dividing $|G|$, then G contains an element of order p.

Hint: use Sylow's theorem to show that G has an Abelian subgroup whose order is dibisible by p.

3. **(a)** Show that if G is a simple group of order < 60, then G is Abelian.
(b*) Bonus. Show that if G is a non-abelian simple group of order < 100, then G is isomorphic to A_5. (Eliminate all orders but 60.)

4. Let G and K be groups. Suppose that there are homomorphisms:

$$\pi : G \to K; \quad s : K \to G,$$

such that the composition πs is the identity automorphism of K. Show that G is isomorphic to a semidirect product of the groups K and H, where H is the kernel of π.

5. Let p be a prime and let G be a group of order p^n. Prove that G is solvable.

6. Prove that G is solvable if and only if one of the following conditions holds:
1) The group G has a filtration

$$G = G_0 \subset G_1 \subset \ldots \subset G_n = \{e\},$$

such that G_i is normal in G_{i-1} and the quotient group G_{i-1}/G_i is commutative.
2) The derived series

$$G = G_0 \subset G_1 \subset G_2 \subset \ldots,$$

where $G_i = [G_{i-1}, G_{i-1}]$, terminates (i.e. has term $G_n = \{e\}$ for some n).