

Algebra I: Homework assignment 6

Due date: October 18

1. Prove that \mathbb{R} is not finitely generated as an Abelian group.
2. Prove that any subgroup of a finitely generated free Abelian group is a free Abelian group.
3. Prove that any discrete subgroup of \mathbb{R}^n is isomorphic to a free Abelian group of rank at most n .
4. Let $L_0 \subset \mathbb{R}^2$ be the standard lattice (spanned by the vectors $(1, 0)$ and $(0, 1)$) and $L \subset \mathbb{R}^2$ be the lattice spanned by the vectors $(1, 2)$ and $(-1, 1)$. Find the quotient group L_0/L .
5. Let G be an Abelian group with generators g_1, g_2, g_3 and relations $g_1 + 2g_2 + 3g_3 = 0$, $2g_1 + 6g_2 + 8g_3 = 0$, $3g_2 + 6g_3 = 0$. Find the decomposition of G into elementary Abelian subgroups.
6. Let L be a sublattice of the standard lattice L_0 in \mathbb{R}^n . Suppose that L is spanned by the rows of an $n \times n$ matrix A with integral entries. Prove that
 - (a) if $\det(A) = 0$, then the quotient group L_0/L is infinite;
 - (b) if $\det(A) \neq 0$, then the quotient group L_0/L is finite, and $|L_0/L| = |\det(A)|$.