

Algebra I: Homework assignment 3

Due date: September 22

1. (a) Find the sign of the following permutation from S_9 :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 7 & 4 & 1 & 2 & 6 \end{pmatrix}.$$

- (b) Prove that the alternating group $A_n \subset S_n$ has index 2 in S_n .

- (c) Prove that A_n is generated by the cycles of length 3.

- (d) Find all conjugacy classes in A_5 , and use this information to show that A_5 is simple.

2. Using Lagrange's Theorem prove Euler's Theorem. Recall that φ denotes the Euler function.

Euler's theorem: If a and n are relatively prime, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

- 3*. Prove that the group $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.

4. (a) Prove that the Klein group $V_4 = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of S_4 . Show that the quotient group S_4/V_4 is isomorphic to S_3 .

- (b) Use the previous result to show the following. To find a root of any polynomial of degree 4 it is enough to find the roots of some cubic polynomial and of two quadratic polynomials.

5. For a set X denote by $\text{Aut}(X)$ the set of all one-to-one maps from X to itself.

- (a) Verify that $\text{Aut}(X)$ is a group under the operation of composition.

- (b) Prove that any group is isomorphic to a subgroup of $\text{Aut}(X)$ for some set X .

6. (**Burnside's lemma**) (a) Let a finite group G act on a set X . Prove that the number of orbits (denoted by $|X/G|$) is equal to the average number of points fixed by an element of G :

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Hint: Show first that

$$\sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|.$$

- (b) Find the number of *different* ways to color the faces of a cube using three colors. Two colorings are *different* if one can not be obtained from the other by a rotation of the cube.