Algebra I: Homework assignment 12
Due date: December 13

1.(About algebras) Let R be a ring containing a field $\mathbb{F}$, i.e. R is an *algebra* over $\mathbb{F}$.

(a) Verify that R is a vector space over $\mathbb{F}$.

(b) Suppose that the dimension of R (regarded as a vector space over $\mathbb{F}$) is finite. Prove that if R is an integral domain, then R is a field.

(c) Give an example of an infinite-dimensional algebra over $\mathbb{Q}$ without zero divisors, which is (i) a field (ii) not a field.

About quadratic integer rings

2. Let d be a square-free integer. Define the *norm* N on the field $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d}\}$ as follows: $N(a+b\sqrt{d}) = (a+b\sqrt{d})(a-\sqrt{d}) = a^2 - db^2$. Define the *trace* of $a+b\sqrt{d}$ to be the sum $(a+b\sqrt{d}) + (a-\sqrt{b}) = 2a$ (this number is the usual trace of the linear operator on $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ coming from the multiplication by $a+b\sqrt{d}$).

Let $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{d})$ be the set of all numbers in $\mathbb{Q}(\sqrt{d})$ whose norm and trace are integer.

(a) Verify that $\mathcal{O}_d$ is a ring. It is called an *integer quadratic ring.*

(b) Prove that

$$\mathcal{O}_d = \begin{cases} \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}, & d \equiv 2, 3 \pmod 4 \\ \{a + b\sqrt{d} : a, b \in \mathbb{Z} \text{ or } a, b \in \frac{1}{2} + \mathbb{Z}\}, & d \equiv 1 \pmod 4. \end{cases}$$

3. Prove that if $d = -1, -2, -3, -7, -11$, then the quadratic integer ring $\mathcal{O}_d$ is a Euclidean domain with respect to the norm $|N|$.

4. Let $L \subset \mathbb{C}$ be a lattice of rank 2 (i.e. the set $\{az + bw : a, b \in \mathbb{Z}\}$, where z and w are two non-collinear vectors in $\mathbb{C}$). Suppose that L is invariant under multiplication by some non-integer complex number α (i.e. $\alpha L = L$). Prove that α is a quadratic integer (i.e belongs to some quadratic integer ring $\mathcal{O}_d$).

5. Show that any ideal in a quadratic integer ring is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ as an Abelian group.

6. Prove that a prime integer p can be represented as $x^2 + 2y^2$ for some integers x and y if and only if -2 is a quadratic residue modulo p.