Algebra I: Homework assignment 11
Due date: December 6

Notation: $\mathbb{F}$ is a field.

1. (Euclidean domains) Prove that R is a Euclidean domain in each of the following cases:
(a) $R = \mathbb{Z}[\sqrt{-2}]$
(b) $R = \mathbb{Z}[\sqrt{3}]$
(c) $R = \mathbb{F}[[x]]$

2. Find a generator for the ideal $(85, 1 + 13i)$ in $\mathbb{Z}[i]$ by the Euclidean algorithm. Do the same for the ideal $(47 - 13i, 53 + 56i)$.

3. Prove that in a Principal Ideal Domain two ideals (a) and (b) are comaximal if and only if a greatest common divisor of a and b is 1.

4. Prove that any two non-zero elements of a P.I.D. have a least common multiple.

5. In each of the following cases, determine whether a is an irreducible element of the ring R. Is a prime?
(a) $R = \mathbb{F}_2[x]$, $a = x^2 + x + 1$;
(b) $R = \mathbb{Z}[\sqrt{5}]$, $a = 2$;
(c) $R = \mathbb{Q}[x]$, $a = x^4 + x^3 + x^2 + x + 1$;
(d) $R = \mathbb{Q}[x]$, $a = x^4 + 4$;

6. Let $f(x) = x^3 + px + q$ be a cubic polynomial with coefficients in the ring $\mathbb{Z}[p, q]$. Show that there is a unique (up to an integer factor) polynomial $D \in \mathbb{Z}[p, q]$ (called *discriminant*) with the following property. For any integers p_0 and q_0, the polynomial $x^3 + p_0x + q_0$ has a multiple root if and only if $D(p_0, q_0) = 0$. Find D explicitly.