Algebra I: Homework assignment 10
Due date: November 29

Notation: $\mathbb{F}$ *is a field.*

1. (About fields) (a) Verify that the set of complex numbers of the form $x + y\sqrt{2}$, where x and y are rational, is a subfield of the field of complex numbers.

(b) Prove that each field of characteristic zero contains a copy of the rational number field.

(c) Prove that the characteristic of a field is always either prime or zero.

(d) Prove that in any field, all proper ideals are trivial.

2. (About rings) Which of the following rings are fields? Integral domains? In each case, find all invertible elements (=*units*) and zero divisors of the ring R.

(a) $R = \mathbb{F}[x]$

(b) $R = \mathbb{Z}[\omega]$, where ω is a primitive cubic root of unity

(c) $R = \mathbb{R}[A]$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(d) $R = \mathbb{R}[A]$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(e) $R = \mathbb{Z}/n\mathbb{Z}$

(f) $R = \mathbb{F}[[x]]$

3. (About ideals) Which of the following rings are the Principal Ideal Domains?

(a) $R = \mathbb{F}[x]$ **(b)** $R = \mathbb{Z}[x]$ **(c)** $R = \mathbb{Z}[\sqrt{-2}]$

(d) $R = \mathbb{Z}[\sqrt{5}]$ **(e)** $R = \mathbb{F}[[x]]$ **(f)** $\mathbb{F}[x, y]$

4. (More about ideals) (a) Find all maximal and all prime ideals in the following rings:

(i) $\mathbb{C}[x]$ (ii) $\mathbb{R}[x]$ (**Bonus**) $\mathbb{C}[x, y]$

(b) Give an example of a non-maximal prime ideal.

(c) Show that in a Principal Ideal Domain all prime ideals are maximal.

5. (a) Prove that a finite integral domain is a field.

(b) (About quotient rings) For each of the quotient rings below answer the following three questions. Is it a field? Is it finite? Is it isomorphic to any ring of Problem 2?

(i) $\mathbb{Z}[i]/(2)$ (ii) $\mathbb{Z}[i]/(1 + i)$ (iii) $\mathbb{Z}[x]/(x^2 + x + 1)$

(iv) $\mathbb{R}[x]/((x - 1)^2)$ (v) $\mathbb{R}[x]/(x^2 + 1)$ (vi) $\mathbb{Z}[x]/(2, x)$

6. Find all primes p such that there exists a non-trivial ring homomorphism $\mathbb{Z}[i] \to \mathbb{Z}/p\mathbb{Z}$.