Applied Algebra: Solutions to Homework 7

1. Explain how to measure 8 units of water using only two jugs, one of which holds precisely 12 units, the other holding 17 units of water.

Solution: We can repeat the following procedure. Fill the larger jug, fill the smaller from the larger, empty the smaller. Then pour the remains from the larger jug to the smaller. If after that the larger jug is not empty, empty the smaller jug and repeat the previuos step. Applying this procedure once we get $(0,5)$ (0 units in the larger jug, 5 units in the smaller). After two procedures we get $(0,17-(12-5))=(0,10)$. After three procedures we get $(0,17-(12-10)-12)=(0,3)$. In general, if we apply procedure to $(0,n)$, we get $(0,5+n)$ if $n \le 7$ and $(0,n-7)$ if $n > 7$ (in other words, each time we take the residue of $5+n$ in division by 12). Hence, we have

$$(0,0) \to (0,5) \to (0,10) \to (0,3) \to (0,8).$$

So to get 8 units in the smaller jug, we apply the procedure 4 times.

2. Find the smallest positive integer number x such that $5x+1$ is divisible by 37.

Solution: First, find an integer solution of the equation $5x+1=37y$. Use Euclidean algorithm or trial and error method whichever is faster.

Euclidean algorithm: 1) $37 = 7\cdot 5+2$ and $2 = 37-7\cdot 5$

2) $5 = 2\cdot 2+1$ and $-1 = 2\cdot(37-7\cdot 5)-5 = 2\cdot 37-15\cdot 5$

Hence, $x=-15, y=-2$ is a solution.

Trial and error method: Find positive integer y such that $37y-1$ is a multiple of 5. By inspection, if $y=3$, then $37y-1$ is a multiple of 5. Hence, $x=22, y=3$ is a solution.

Any other solution has form $(22+37k, 3-5k)$, $k\in\mathbb{Z}$ (e.g. the solution $(-15,-2)$ found by Euclidean algorithm is obtained when $k=-1$). Hence, $(22,3)$ is the smallest positive solution.

Answer: $x=22$

3. Find the remainder of $1,000,000,000,000$ in division by 21.

Solution 1: First, $1,000,000,000,000 = 10^{12} = 100^6$. Since $100 \equiv (-5) \pmod{21}$ and $5^2 = 25 \equiv 4 \pmod{21}$, we have $100^6 \equiv (-5)^6 = 5^6 \equiv 4^3 = 64 \equiv 1 \pmod{21}$.

Solution 2: Since 10 and 21 are relatively prime, $10^{\varphi(21)} \equiv 1 \pmod{21}$ by Euler's theorem. Compute $\varphi(21)=\varphi(3)\varphi(7)=2\cdot 6=12$. Hence, $1,000,000,000,000 = 10^{12} \equiv 1 \pmod{21}$.

Answer: 1

4. Find all integer numbers x such that $x-5$ is divisible by 7 and $x-4$ is divisible by 11.

Solution: We want to find all x such that $x-4=11y$ and $x-5=7z$ for some integers y and z. Subtracting the second equation from the first, we get the equation $1=11y-7z$. It has a solution $y = 2$, $z = 3$. Then $x = 11 \cdot 2 + 4 = 26$. The other solutions of the initial system have form $26 + LCM(7, 11)k = 26 + 77k$ where k is any integer.

Answer: $x = 26 + 77k$, $k \in \mathbb{Z}$

5. (a) Are there any numbers n such that $\varphi(n) = 14$? Explain!
(b) Find all n such that $\varphi(n) = 16$.

Solution: (a) No. Prove this by contradiction. Let $n = p_1^{k_1} \dots p_l^{k_l}$ be the prime decomposition for n. Then $14 = \varphi(n) = (p_1 - 1)p_1^{k_1 - 1} \dots (p_l - 1)p_l^{k_l - 1}$. In particular, if p is a prime divisor of n, then $p - 1$ divides $\varphi(n)$. Hence, $p - 1 = 1, 2, 7$ or 14. Since neither 8 nor 15 is prime, the only prime divisors of n are 2 and 3. But if $n = 2^k$, then $\varphi(n) = 2^{k-1} \neq 14$. If $n = 2^k 3^m$, and $m > 0$, then $\varphi(n) = 2^k 3^{m-1} \neq 14$.

(b) Again we use that if p is a prime divisor of n, then $p - 1$ divides $\varphi(n)$. The divisors of 16 are $1, 2, 4, 8$ and 16. All of them except for 8 have form $p - 1$ for prime numbers $2, 3, 5$ and 17. Since 16 is not a multiple of 3, 5 or 17, these primes can enter into the decomposition of n only once. Also, if $n = 17k$ and k is relatively prime with n, then $\varphi(n) = 16\varphi(k)$. Hence, $k = 1$ or 2, and $n = 17$ or 34. The remaining cases are as follows: $n = 2^k$, $n = 2^k \cdot 3$, $n = 2^k \cdot 5$, $n = 2^k \cdot 3 \cdot 5$. In each case we have the equation on k: $16 = \varphi(n) = 2^{k-1}$, $16 = 2^k$, $16 = 2^{k-1} \cdot 4$, $16 = 2^k \cdot 4$. Solving them for k we get that $n = 32, 48, 40$ and 60.

Answer: $n = 32, 48, 40, 60, 17, 34$