Applied Algebra: Homework assignment 2
Due date: September 15

Mathematical induction:

1. Prove that for all positive integers n,

$$1 + 4 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Find all positive integers n such that $3^n + 2^n$ is divisble by 5. Use mathematical induction to prove your answer.

Divisibility criteria:

3. Let $\overline{a_9 a_8 \ldots a_0}$ be the decimal notation for a positive ten digit integer a (i.e. a_0, a_1,…, a_9 are digits such that $a = a_9 10^9 + a_8 10^8 + \ldots + a_0$).

(a) Prove the following divisibility criterion for 99:

The ten digit number a is divisible by 99 iff the sum of 2-digit numbers

$$\overline{a_1 a_0} + \overline{a_3 a_2} + \overline{a_5 a_4} + \overline{a_7 a_6} + \overline{a_9 a_8}$$

is divisible by 99.

(b) For 10 digit numbers, formulate a divisibility criterion by 101.

Prime numbers:

4. Find the prime factorizations for the following integers:

(a) 1001; **(b)** 999; **(c)** 150; **(d)** 136.

5. (a) Show that if $2^n - 1$ is prime, then n must be prime.

Hint: use the formula for the sum of a geometric progression:

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1.$$

Remark: prime numbers of the form $2^p - 1$ (where p is prime) are called Mersenne numbers.

(b) Show that $2^{11} - 1$ is not prime.

Bonus 6. Prove that there are infinitely many primes of the form $4k + 3$ where k is integer.

Hint: Repeat Euclid's proof in this case and see what happens with the residues modulo 4.