Section 3.2

1) a) $a \sim b$ is true because $b = a \circ c$.

b) $a \sim c$ is false.

c) $b \sim e$ is false.

d) $b \sim c$ is false.

3) The left cosets of H are as follows:

$$\begin{aligned}(0,0)+H &= \{(0,0),(1,2)\}\\ (0,1)+H &= \{(0,1),(1,3)\}\\ (0,2)+H &= \{(0,2),(1,0)\}\\ (0,3)+H &= \{(0,3),(1,1)\}\end{aligned}$$

6) By definition, $x \sim y$ if and only if $y = x \circ h$ for some $h \in H$. But this happens if and only if $x^{-1} \circ y = x^{-1} \circ (x \circ h) = (x^{-1} \circ x) \circ h = e \circ h = h$ for some $h \in H$. Therefore, $x \sim y \iff x^{-1} \circ y \in H$.

7) For each part, we only need to compute $x^{-1} \circ y$ and check if it is in H.

a) $(41)(1342) = (13)(24) \in H$. Yes.

b) $(321)(234) = (134) \notin H$. No.

c) $(43)(21)(23) = (1243) \notin H$. No.

d) $(41)(123) = (1234) \in H$. Yes.

9) a) For a group G and a subgroup H, the right coset of an element $x \in G$ is defined by $Hx = \{h \circ x | h \in H\}$.

b) We can define the equivalence relation by $x \sim y$ if and only if $y = h \circ x$ for some $h \in H$.

c) The right coset decomposition is

$$S_3 = \{e,(12)\} \cup \{(13),(132)\} \cup \{(23),(123)\}$$

d) No.

12) We know that a code word from H has been transmitted, and the word g has been received. Therefore, if h is transmitted, then the error that occurs is $g - h$, which is the same as $g + h$ in $(\mathbb{Z}_2)^n$. Therefore, the set of possible errors that could have occurred is $\{g + h | h \in H\} = g + H$. The most likely errors are the ones of smallest weight; i.e. the coset leaders.

Section 3.3

1) a) $X_e = \{1,2,3\}$; $X_{(12)} = \{3\}$; $X_{(13)} = \{2\}$; $X_{(23)} = \{1\}$; $X_{(123)} = X_{(132)} = \emptyset$.
$G_1 = \{e, (23)\}$; $G_2 = \{e, (13)\}$; $G_3 = \{e, (12)\}$.

b) $X_e = \{1,2,3,4,5,6\}$; $X_{(12)} = \{3,4,5,6\}$; $X_{(345)} = X_{(354)} = \{1,2,6\}$; $X_{(12)(345)} = X_{(12)(354)} = \{6\}$.
$G_1 = G_2 = \{e, (345), (354)\}$; $G_3 = G_4 = G_5 = \{e, (12)\}$;
$G_6 = \{e, (12), (345), (354), (12)(345), (12)(354)\}$.

2) a) i) The only equivalence class is $X = \{1,2,3\}$.

ii) For any x, $|[x]| = 3$, and $|G_x| = 2$. Therefore, $|G| = 6 = 3 \cdot 2$ checks.

b) i) The equivalence class decomposition is $X = \{1,2\} \cup \{3,4,5\} \cup \{6\}$.

ii) For $x \in \{1,2\}$, $|[x]| = 2$ and $|G_x| = 3$. For $x \in \{3,4,5\}$, $|[x]| = 3$ and $|G_x| = 2$. For $x = 6$, $|[x]| = 1$ and $|G_x| = 6$. In any case, $|[x]||G_x| = 6 = |G|$, as desired.