Section 2.6

1) a) Yes.

b) No. Two elements sent to C.

c) Yes.

d) No. Two elements sent to A.

2) **a)**

$$\begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix}$$

d)

$$\begin{pmatrix} A & B & C & D & E & F \\ D & F & A & B & E & C \end{pmatrix}$$

3) **a)**

$$\begin{pmatrix} A & B & C & D & E \\ D & A & B & E & C \end{pmatrix}$$

c)

$$\begin{pmatrix} A & B & C & D & E & F \\ C & D & E & F & B & A \end{pmatrix}$$

4) **a)** Not a permutation group. For instance:

$$\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

which is not in the set.

c) This is a permutation group.

5) a) (ACD)

b) $(AC)(BE)$

c) $(BDG)(CFE)$

d) $(AD)(BEFG)(CH)$

6) a)

$$\begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}$$

b)

$$\begin{pmatrix} A & B & C & D & E \\ C & D & A & E & B \end{pmatrix}$$

c)

$$\begin{pmatrix} A & B & C & D & E \\ D & E & C & A & B \end{pmatrix}$$

d)

$$\begin{pmatrix} A & B & C & D & E & F \\ A & C & D & B & F & E \end{pmatrix}$$

7) a) (ADEC)

b) (BDF)

Section 3.1

1) a) x is a sister of y is NOT reflexive, NOT symmetric, and NOT transitive. (It is not reflexive because no one is their own sister. It is not symmetric because it is possible for boys to have sisters. They would be the sister's brother, not the sister's sister. To see that it is not transitive, suppose that x and y are sisters. Then $x\ y$ and $y\ x$. If the relation were transitive, it would force x to be her own sister.

b) x is a first cousin of y is NOT reflexive, and NOT transitive. But it is symmetric. Again, the claim that it is not transitive probably needs the most explanation. Suppose x and y are first cousins on y's mother's side, and suppose y and z are first cousins on y's father's side. It does not follow that x and z are first cousins. Another possibility is that x and z are both cousins of y on the same side of the family, but x and z are actually siblings and not cousins.)

c) x and y have a common ancestor is reflexive and symmetric. Whether or not it is transitive is open to philosophical debate. The answer is 'yes' if you believe that everyone has a common ancestor. The answer is probably 'no' otherwise. For instance, x and y could have a common ancestor on y's mother's side, while y and z could have a common ancestor on y's father's side. Then x and z are related by marraige, but do not necessarily have a blood relative in common.

d) x is a descendant of y is NOT reflexive, NOT symmetric, but it is transitive.

2) a) $X = \{1, 2\} \cup \{3, 4\}$

b) $X = \{1\} \cup \{2, 3, 4\} \cup \{5\}$

c) $X = \{1, 2, 3\} \cup \{4, 5\} \cup \{6\}$

3) The three equivalence classes are: $\{3n | n \in \mathbb{Z}\}$, $\{3n + 1 | n \in \mathbb{Z}\}$, and $\{3n + 2 | n \in \mathbb{Z}\}$.

8) We have to show that the equivalence classes that the group $\langle g \rangle$ induce on X are precisely the cycles in the cycle decomposition of g. To see this, first let us say what it means for A and B in X to be in the same cycle of g. It means that we can get from A to B by applying g a certain number of times: $g^i(A) = B$. But g^i is an element of the group $\langle g \rangle$, so $g^i(A) = B$ means that A and B are in the same $\langle g \rangle$ equivalence class.

9) a) For any $x \in G$, $e \circ x \circ e^{-1} = x$, so conjugacy is reflexive. If $x\ y$, then there exists $g \in G$ such that $g \circ x \circ g^{-1} = y$. Solving for x, we see $x = g^{-1} \circ x \circ g = g^{-1} \circ x \circ (g^{-1})^{-1}$. Thus, $y\ x$ and so conjugacy is symmetric.

Now suppose $x\ y$ and $y\ z$. Then there exist g and h in G such that $gxg^{-1} = y$ and $hyh^{-1} = z$. Therefore, $z = h(gxg^{-1})h^{-1} = (hg)x(g^{-1}h^{-1}) = (hg)x(hg)^{-1}$. So $x\ z$ and conjugacy is transitive.

b) First, conjugacy is an equivalence relation, so $x \in [x]$. Suppose now that $y \in [x]$, i.e. $x\ y$. Then there exists $g \in G$ such that $gxg^{-1} = y$. Since G is commutative, we have $y = gxg^{-1} = xgg^{-1} = x$. Therefore, $[x] = \{x\}$.

c) In cycle notation, the conjugacy class decomposition of S_3 is

$$S_3 = \{e\} \cup \{(123), (132)\} \cup \{(12), (13), (23)\}.$$

10) a) For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$, so the relation is reflexive.

If $x \equiv y (\mod 1)$, then $x - y \in \mathbb{Z}$. But then $y - x = -(x - y) \in \mathbb{Z}$. Therefore, $y \equiv x (\mod 1)$ and the relation is symmetric.

Now suppose $x \equiv y (\mod 1)$ and $y \equiv z (\mod 1)$. Then $x - y$ and $y - z$ are integers. Therefore, $x - z = (x - y) + (y - z)$ is an integer, so $z \equiv z (\mod 1)$. The relation is transitive.

b) Any real number x can be written uniquely as $n + r$, where n is an integer and r, the remainder or fractional part, satisfies $0 \leq r < 1$. Suppose $x = n + r$ and $y = m + s$, where $n, m \in \mathbb{Z}$ and r, s are the fractional parts of x, y, respectively. It follows that $-1 < r - s < 1$. If $x \equiv y (\mod 1)$, then $x - y = (n - m) + (r - s)$ is an integer. Since $n - m$ is an integer. This forces $r - s$ to be an integer. Because $-1 < r - s < 1$, $r - s$ must be 0. It follows taht $x \equiv y (\mod 1)$ if and only if their fractional parts are equal.