Section 2.4

1) a) h_3

b) h_7

c) h_2

d) h_5

2) **a)** $h_3 \circ h_8 = h_7$; $(h_3 \circ h_8)^{-1} = h_7^{-1} = h_7$. $h_3^{-1} = h_3$; $h_8^{-1} = h_8$; $h_8^{-1} \circ h_3^{-1} = h_8 \circ h_3 = h_7$.

c) $2 + 3 = 5$; $-(2 + 3) = -5 = 1$. $-2 = 4$; $-3 = 3$; $-3 + -2 = 3 + 4 = 1$.

3d) $(a^{-1} \circ b^{-1}) \circ (c^{-1} \circ a)^{-1} \circ c^{-1} = a^{-1} \circ b^{-1} \circ a^{-1} \circ c \circ c^{-1} = a^{-1} \circ b^{-1} \circ a^{-1}$.

5)

$$\begin{array}{rcll} x \circ a & = & b & \\ (x \circ a) \circ a^{-1} & = & b \circ a^{-1} & \text{compose both sides with } a^{-1} \\ x \circ (a \circ a^{-1}) & = & b \circ a^{-1} & \text{associativity of the group operation} \\ x \circ e & = & b \circ a^{-1} & \text{definition of } a^{-1} \\ x & = & b \circ a^{-1} & \text{definition of identity} \end{array}$$

7) $(a \circ b) \circ f = e \circ f = a$, while $a \circ (b \circ f) = a \circ e = f$, and so the axiom of associativity fails. However, the conclusion of Theorem 2.8 still holds since every element of the set appears exactly once in each row and column.

9)

$$\begin{array}{rcll} x \circ a & = & y \circ a & \\ (x \circ a) \circ a^{-1} & = & (y \circ a) \circ a^{-1} & \text{compose both sides with } a^{-1} \\ x \circ (a \circ a^{-1}) & = & y \circ (a \circ a^{-1}) & \text{associativity of the group operation} \\ x \circ e & = & y \circ e & \text{definition of } a^{-1} \\ x & = & y & \text{definition of identity} \end{array}$$

10) Left Cancellation Law: Let $(G, \circ)$ be a group. Let x, y, and a be group elements such that $a \circ x = a \circ y$. Then $x = y$.

$$\begin{array}{rcll} a \circ x & = & a \circ y & \\ a^{-1} \circ (a \circ x) & = & a^{-1} \circ (a \circ y) & \text{compose both sides with } a^{-1} \\ (a^{-1} \circ a) \circ x & = & (a^{-1} \circ a) \circ y & \text{associativity of the group operation} \\ e \circ x & = & e \circ y & \text{definition of } a^{-1} \\ x & = & y & \text{definition of identity} \end{array}$$

12) Let $(G, \circ)$ be a group with two elements. Let us denote the elements by e and a, where e is the identity. We know that $e \circ e = e$, $e \circ a = a$ and $a \circ e = a$. Thus, in order for G to be a group, we must have $a \circ a = e$. In other words, the composition table must be the following:

$\circ$	e	a
e	e	a
a	a	e

Therefore, every two element group is isomorphic, with the isomorphism determined by identifying the identities with each other and idetifying the nonidentity elements with each other.

Section 2.5

1) a) No, for instance $1 + 3 = 4$.

b) Yes.

c) No, for instance $h_2 \circ h_3 = h_4$.

d) Yes

2) **a)** No; the subset is not closed under the operation: $(1, 2) + (1, 2) = (0, 1)$.

b) Yes.

3) **a)** The order of 1 in $\mathbb{Z}_6$ is 6. $\langle 1\rangle = \{0, 1, 2, 3, 4, 5\}$.

c) The order of 3 in $\mathbb{Z}_6$ is 2. $\langle 3\rangle = \{0, 3\}$.

e) The order of 0 in $\mathbb{Z}_4$ is 1. $\langle 0\rangle = \{0\}$.

g) The order of h_5 in Table 2.1 is 2. $\langle h_5\rangle = \{h_1, h_5\}$.

7) $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a cyclic group because all of its elements is its own inverse. Therefore, any non-identity element has order 2. It cannot generate all 4 elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a cyclic subgroup.

8) If H is a subgroup of G that contains g, then H must also contain g^2, g^3, etc, because H is closed under the group operation of G. Furthermore, H contains inverses, so it must contain g^{-1}, g^{-2}, g^{-3}, etc. We know that it must contain the identity, as well. In summary, $g^n \in H$ for any integer n. Therefore, $\langle g\rangle \subseteq H$, and it is in this sense that $\langle g\rangle$ is the smallest subgroup containing g.

10) For K to be a subgroup of H, it means that K is a group under the composition law of H. But this is the same composition law in G because H is a subgroup of G. Therefore, K is a group under the composition law of G; i.e. a subgroup of G.

11) First, we show that $H \cap K$ is closed under $\circ$. Let a and b be arbitrary elements in $H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H and K are both subgroups, they are closed under $\circ$, so that $a \circ b \in H$ and $a \circ b \in K$. Therefore, $a \circ b \in H \cap K$, and $H \cap K$ is closed under $\circ$.

According to Theorem 2.10, the only other condition we must meet is that $H \cap K$ is closed under inversion. Let $s \in H \cap K$. We must show that $s^{-1} \in H \cap K$. But $s \in H$ and $s \in K$, and since they are both subgroups, we have $s^{-1} \in H$ and $s^{-1} \in K$. This yields the desired conclusion.

12a) $\langle 2\rangle = \{0, 2, 4\}$, and $\langle 3\rangle = \{0, 3\}$. It is clear that $\langle 2\rangle \cup \langle 3\rangle = \{0, 2, 3, 4\}$ is not a subgroup of $\mathbb{Z}_6$ since it is not closed under addition: $2 + 3 = 5$.