MAT 312/AMS 351 – Fall 2010
Homework 11

1. A *linear fractional transformation* $f(x)$ is a function of the form
$$f(x) = \frac{ax+b}{cx+d}$$
where a, b, c, d, are real numbers satisfying $ad - bc = 1$. Show that if $f(x)$ as above and $g(x) = (ex + f)/(gx + h)$ are linear fractional transformations, so is their composition $f \circ g$ (where $f \circ g(x) = f(g(x))$ as usual). Hint: the coefficients in $f \circ g$ are related to those in f and g by matrix multiplication, where the matrices have determinant 1.

2. Show that in $\mathbf{Z}_{13}^*$ the elements $\{1, 3, 4, 9, 10, 12\}$ form a subgroup. Call it H. Show that $H =< 4 >$, i.e. that the elements $4^0 = 1, 4^1 = 4, 4^2 = 3$, etc. make up all of H. Conclude that H is isomorphic to the additive group $\mathbf{Z}_6$. Explain carefully.

3. For every divisor of $|\mathbf{Z}_{24}| = 24$, identify a subgroup of $\mathbf{Z}_{24}$ with that cardinality. (This should be easy).

4. For every divisor of $|S(4)| = 24$, identify a subgroup of $|S(4)|$ with that cardinality. (Not so easy. Hints can be found on pages 82 and 92 in Kra.)

5. Let $GL(2, \mathbf{Z}_2)$ represent the group of 2×2 matrices with coefficients in $\mathbf{Z}_2$ and determinant 1 (calculated mod 2). What is the cardinality n of this group? List its elements. Write out its multiplication table. Identify the group with a subgroup of $S(n)$ (as usual, thinking of each element as defining, by left-multiplication, a permutation of the elements of the group). Is this a group we have seen before, perhaps in different clothing? *Hint:* consider the action by left-multiplication of $GL(2, \mathbf{Z}_2)$ on the column vectors
$$\mathbf{1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

6. In the permutation group $S(4)$, let H represent the subgroup
$$H = \{e, (1234), (13)(24), (1432)\}.$$
H should have 6 left cosets. What are they? Describe them by listing their elements.