Math 310: Homework 3

If $v_1, \dots, v_n$ are elements if a vectorspace V, we will use the notation $\mathrm{sp}(v_1, \dots, v_n)$ for the subspace they span.

Ex 1. (i) Let V be a real vector space with basis v_1, v_2, v_3. Construct a linear map $T : V \to W = \mathbb{R}^2$ that is surjective and has the property that $T(v_1 + v_2 - 3v_3) = 0$.

(ii) Is it possible to construct a linear map with these properties if $W = \mathbb{R}^3$? Give an example, or explain why not.

Ex 2. (i) Let U be a subspace of V, and suppose that $T : U \to W$ is a linear map. Show that it is always possible to extend T to a linear map $T' : V \to W$. i.e. show that there is a linear map $T' : V \to W$ such that $T'(u) = T(u)$ for all $u \in U$.

(ii) Suppose that $U = \mathrm{sp}(e_1, e_2) \subset \mathbb{R}^5 = V$, $W = \mathbb{R}^4$ and that $T : U = \mathbb{R}^2 \to \mathbb{R}^4$ is given by

$$T(e_1) = \sum_{j=1}^{4} e_j, \qquad T(e_2) = e_1.$$

(Here $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$; see p 27.)

(a) Since T is a map $\mathbb{R}^2 \to \mathbb{R}^4$, it is given by multiplication by a matrix A. Write down this matrix A.

(b) Write down a matrix for T'. Choose this matrix so that T' is injective (if possible) and surjective (if possible). Explain your answer.

(iii) Now go back to the general problem in (i). Under what conditions on U, V, W, T can you choose T' to be injective? Under what conditions can you choose T' to be surjective? Give the most general conditions you can find.

Ex 3. Suppose that $T : V \to W$ is a linear map and that $v_1, \dots, v_n$ is a basis for V. Suppose that the list $Tv_1, \dots, Tv_n$ is linearly dependent in W. Show that T is not injective.
Note: For this question, you may use any result in the book up to and including Theorem 3.4.

Ex 4. (i) Prove that there does not exist a linear map $T : \mathbb{R}^6 \to \mathbb{R}^2$ with null space equal to

$$\{(x_1, \dots, x_6) : x_1 + x_2 + x_3 = 0, x_2 = -x_4 = x_6\}.$$

(ii) Give the matrix of a linear map $T : \mathbb{R}^6 \to \mathbb{R}^2$ with null space

$$\{(x_1, \dots, x_6) : x_1 + x_2 + x_3 = 0, x_2 = -x_4\}.$$

We saw in class that the space $\mathcal{L}(V, W)$ of linear maps from V to W is always a vector space. Take $W = \mathbb{F}$. We then get the space $V^* := \mathcal{L}(V, \mathbb{F})$ of $\mathbb{F}$-linear maps

$V \to \mathbb{F}$. This is called the **dual space** of V. The next two exercises ask you to explore its structure.

Ex 5. Let $V = \mathbb{F}^2$ with basis e_1, e_2. Define elements $e_1^*, e_2^* \in V^*$ by:

$$e_1^*(e_1) = 1,\ e_1^*(e_2) = 0, \qquad e_2^*(e_1) = 0,\ e_2^*(e_2) = 1.$$

Show that e_1^*, e_2^* form a basis for V^*. Deduce that $\dim(\mathbb{F}^2)^* = 2$.

Bonus ex 6: (i) Show that if V is a vector space of dimension n then V^* also has dimension n.

(ii) If V has infinite dimension then so does V^*. However, even if we have a basis for V it is not easy to define a basis for V^*. For example suppose that V is the set of infinite sequences that are eventually 0:

$$V := \{(x_1, x_2, x_3, \dots) : x_i \neq 0 \text{ for only finitely many } i\}.$$

Then V has the basis $e_i, i \in \mathbb{N}$, where e_i has 1 in the ith place and zeros elsewhere. As before we can define $e_i^* \in V^*$ which equals 1 on e_i and 0 on all other e_j. Find an element of V^* that is NOT in $\mathrm{sp}(e_1, e_2, e_3, \dots)$.