MAT 310 Linear Algebra Midterm 2 Solutions

1. Write a complete, clear and correct proof of the following statement: Given v_1 and v_2 eigenvectors of a linear operator; if their associated eigenvalues are different, then v_1 and v_2 are linearly independent.

   Solution. (This is an abridged version of a proof in the book). Let $\lambda_1 \neq \lambda_2$ be the corresponding eigenvalues. If T is the operator, then $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$. Suppose a linear relation $a_1v_1 + a_2v_2 = 0$. Since v_1 and v_2, being eigenvectors, are not zero, both a_1 and a_2 are nonzero. Then for example $v_1 = -\frac{a_2}{a_1}v_2$. so $Tv_1 = -\frac{a_2}{a_1}Tv_2 = -\frac{a_2}{a_1}\lambda_2 v_2$; on the other hand, $Tv_1 = \lambda_1 v_1 = -\frac{a_2}{a_1}\lambda_1 v_2$. Since everything else is nonzero, λ_1 must equal λ_2, a contradiction. So there is no such linear relation, and v_1, v_2 are linearly independent.

2. Consider the linear operator T defined in the standard basis $(1,0), (0,1)$ by the matrix
$$\begin{bmatrix} 11 & -4 \\ 30 & -11 \end{bmatrix}.$$
   Take $v = (1,0)$ and note that $T(v) = (11,30)$ and $T^2(v) = (1,0)$, so T satisfies the equation $(T^2 - I)v = 0$. Factor $T^2 - I$ as $(T - aI)(T - bI)$. When you have calculated a and b, apply $(T - bI)$ and then $(T - aI)$ to v to obtain an eigenvector for T, as in the proof in the text that every operator on a complex vector space has an eigenvector.

   Solution. The polynomial $x^2 - 1$ factors as $(x+1)(x-1)$, so $T^2 - I = (T+I)(T-I)$, giving $a = -1$, $b = 1$ in the statement of the problem. If we apply
$$T - I = \begin{bmatrix} 10 & -4 \\ 30 & -12 \end{bmatrix}$$
   to $v = (1,0)$ we get $(10,30)$. This is an eigenvector, with eigenvalue -1 as can be checked. (If we applied
$$T + I = \begin{bmatrix} 12 & -4 \\ 30 & -10 \end{bmatrix}$$
   instead, we would get $(12,30)$, an eigenvector with eigenvalue 1.) Similar analysis for the other form of this problem, where $v = (0,1)$.

3. Give an example of a linear operator on $\mathbf{R}^2$ which is not diagonalizable but can be put in upper- triangular form.

Solution. Suppose T is an operator which with respect to some basis is in upper-triangular form, with matrix

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}.$$

We know that the diagonal entries are eigenvalues of T, and that if they are distinct the corresponding eigenvectors are linearly independent; using those eigenvectors as basis gives a diagonal matrix. So a and b must be equal. If $c = 0$ then our matrix is diagonal. So take for example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We check this matrix is not diagonalizable. The only eigenvalue is 1; a possible eigenvector must satisfy

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The first components give $x + y = x$ so $y = 0$. So any eigenvector is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. There cannot be two linearly independent vectors, so this transformation is not diagonalizable.