MAT 200
SOLUTIONS TO HOMEWORK 4

OCTOBER 5, 2004

Section 3.4: Problem 5

(5) (a) True, all numbers have a unique square.
(b) False, for the same reason as before.
(c) False, for the same reason as before.
(d) False, for the same reason as before.
(e) False, $x^2 + y^2$ has both a positive and a negative square root.
(f) True, $x = -1$ is the only possible choice for x.

Section 4.3: 5, 14, 17*

(5) (a) n is divisible by k:

$$\exists l\ (lk = n)$$

For future use, we denote this statement by $k|n$

(b)

(1)	$k\|n$	Premise
(2)	$\exists l\ (lk = n)$	Definition of symbol $\|$
(3)	$ck = n$	Existential Specification of (2)
(4)	$ckn = n^2$	from (3)
(5)	$(cn)k = n^2$	from (4)
(6)	$\exists l\ (lk = n^2)$	Existential Generalization of (5)
(7)	$k\|n^2$	(6), Definition of $\|$

(c)

(1)	$k\|n - 1$	Premise
(2)	$\exists l\ (lk = n - 1)$	Definition of $\|$
(3)	$ck = n - 1$	Existential Specification of (2)
(4)	$n^2 - 1 = (n + 1)(n - 1)$	Explicit calculation
(5)	$(n + 1)ck = n^2 - 1$	(3), (4)
(6)	$\exists l\ (lk = n^2 - 1)$	Existential Generalization of (5)
(7)	$k\|n^2 - 1$	Definition of $\|$

(d) Certainly $6^2 = 36$ is a multiple of 36, but 6 is not a multiple of 36. This is a counterexample to $k|n^2 \to k|n$, with $n = 6$ and $k = 36$.

(14) Consider the domain of the integers. Define $P(x)$ to be true iff x is odd, and define $Q(x)$ to be true iff x is even. We apply this example to the converse of theorem 4.6 (b) and (d) as below:

(b) The converse to theorem 4.6 (b) is $\forall x\ [P(x) \vee Q(x)] \to (\forall x\ P(x)) \vee (\forall x\ Q(x))$. In this case, $\forall x\ [P(x) \vee Q(x)]$ is the statement that "all integers are even or odd", which is true. The statement that $(\forall x\ P(x)) \vee (\forall x\ Q(x))$ states "all integers are even or all integers are odd", which is clearly false. Thus we have found a counterexample to the converse of theorem 4.6 (b).

(d) The converse to theorem 4.6 (d) is $\Big(\exists x\ P(x) \wedge \exists x\ Q(x)\Big) \to \exists x\ [P(x) \wedge Q(x)]$. In this case, $\exists x\ P(x) \wedge \exists x\ Q(x)$ is the statement that "there is an even integer and there is an odd integer", which is true. The statement that $\exists x[P(x) \wedge Q(x)]$ states "there exists an integer which is both even and odd", which is clearly false. Thus we have found a counterexample to the converse of theorem 4.6 (d).

(17) In this proof, the mistake is to apply EG to the entire statement $\forall y(y + (3 - y) = 3)$ to yield $\exists x\ \forall y\ (y + x = 3)$. The problem here is that x is dependent on y, namely $x = 3 - y$. More formally, plugging in $3 - y$ for x in $P(x) = \forall y\ (y + x = 3)$ is not allowed since it leads to a conflict of variables — see Note 3 in the handout. In fact, the resulting statement is false.

Section 4.4: 10, 27, 30

(10) We wish to show: $\sim\exists x\ (0 \cdot x = 1)$. We prove this by contradiction. Suppose there is an x for which $0 \cdot x = 1$. But we also know that for any x, $0 \cdot x = 0$ (this is Theorem A-5 in Appendix 2). By transitivity of equality, we get $0 = 1$, which contradicts one of the axioms of real numbers (Axiom V-12 on page 375).

This proof is not quite formal but can be turned into a formal proof with little effort.

(27) (a) Assume $x < y$. By Axiom V-16 on page 375, this implies $x + z < y + z$ for any z. Take $z = -x - y$ (this implicitly uses US rule). Then we get $x + (-x - y) < y + (-x - y)$. Using commutativity and associativity of addition we get $(x - x) - y < (y - y) - x$. By definition, $x - x = 0, y - y = 0$, so we get $-y < -x$, which is the same as $-x > -y$ (this is the definition of $>$).

Now assume $-x > -y$. Add to both sides $z = x + y$ (this is a short way of saying: By Axiom V-16 on page 375, this implies $-x + z > -y + z$ for any z. Take $z = x + y$). We get $-x + (x + y) > -y + (x + y)$. Using commutativity, associativity of addition and $-x + x = 0, -y + y = 0$, we get $y > x$, which is the same as $x > y$.

Thus, we have shown that $(x < y) \to (-x > -y)$ and $(-x > -y) \to x < y$, so $(x < y) \leftrightarrow (-x > -y)$ (by the biconditional rule).

(b) By applying UG rule to part (a), we get $\forall x, y\ (x < y) \leftrightarrow (-x > -y)$. In particular, it should hold for $x = 0$ (by US rule), so we get $\forall y\ (0 < y) \leftrightarrow (-y > -0)$. Since $-0 = 0$, we get $\forall y\ (0 < y) \leftrightarrow (-y > 0)$.

(30) Assume $0 < x < y$. By Axiom V-17 on page 375, we can multiply both sides of inequality $x < y$ by any positive number z. Take $z = x$ (since we know that x is positive); then we get $x^2 < xy$. Similarly, by transitivity of $<$ (Axiom V-14), we know that $0 < y$, so we can multiply $x < y$ by y to get $xy < y^2$. Thus, we have $x^2 < xy$ and $xy < y^2$. By transitivity of $<$, this implies $x^2 < y^2$.