Stony Brook University MAT 127 Calculus C

Power series solutions for initial-value problems.

The Method. Power series give a very general method for solving initial-value problems. For convenience in notation, we will take the initial value or values to be given at $x = 0$.

An initial-value problem that we cannot solve with this term's methods is

$$y'' + y = \sin x$$
$$y(0) = 0, \quad y'(0) = 0$$

(This problem models a harmonic oscillator with period 2π starting from rest and being driven at its natural frequency).

The power series method consists in taking y to be given by an unknown power series in x:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

and then using the equation and the initial conditions to solve for the coefficients $c_0, c_1, c_2, \dots$. Here it is immediate that $c_0 = y(0) = 0$, and that $c_1 = y'(0) = 0$, but the equation will force relations between the cs which allow them all to be calculated.

Differentiating the power series twice, term by term, we get:

$$\begin{aligned} y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots \\ y'' &= 2c_2 + 2 \cdot 3c_3 x + 3 \cdot 4c_4 x^2 + 4 \cdot 5c_5 x^3 + 5 \cdot 6c_6 x^4 + \dots \ . \end{aligned}$$

Now we can write $y'' + y$ as a single power series by adding the coefficients of like powers as usual:

$$\begin{aligned} y'' + y &= 2c_2 + c_0 + (2 \cdot 3c_3 + c_1)x + (3 \cdot 4c_4 + c_2)x^2 + \\ &\quad (4 \cdot 5c_5 + c_3)x^3 + (5 \cdot 6c_6 + c_4)x^4 + \dots . \end{aligned}$$

We know $\sin x$ has the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

so our differential equation becomes an equation between two power series. The two sides of the equation can be equal for all values of x only if all like coefficients on left and right are equal. This gives a system of equations:

$$\begin{aligned}
2c_2 + c_0 &= 0 \\
2 \cdot 3c_3 + c_1 &= 1 \\
3 \cdot 4c_4 + c_2 &= 0 \\
4 \cdot 5c_5 + c_3 &= -\frac{1}{3!} \\
5 \cdot 6c_6 + c_4 &= 0 \\
6 \cdot 7c_7 + c_5 &= \frac{1}{5!} \\
&\cdots
\end{aligned}$$

We already know $c_0 = 0$ and $c_1 = 0$ from our initial conditions. Furthermore the equations for even indices all have right-hand side 0; and since $c_0 = 0$, it follows that $c_2 = 0$, so $c_4 = 0$, ... All the even-indexed coefficients are 0. For the odd coefficients we may solve the system recursively:

$$\begin{array}{rclcrcl}
 & & & & c_1 & = & 0 \\
2 \cdot 3c_3 + c_1 & = & 1 & \text{so} & c_3 & = & \frac{1}{3!} \\
4 \cdot 5c_5 + c_3 & = & -\frac{1}{3!} & \text{so} & c_5 & = & -\frac{2}{5!} \\
6 \cdot 7c_7 + c_5 & = & \frac{1}{5!} & \text{so} & c_7 & = & \frac{3}{7!} \\
 & & & & \cdots & &
\end{array}$$

This pattern will continue, as can easily be checked, giving

$$c_1 = 0, \qquad c_{2n+1} = (-1)^{n+1} \frac{n}{(2n+1)!} \text{ for } n \geq 1.$$

It follows from the ratio test that the function given by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

with these coefficients is defined for all values of x. It is the solution to our initial value problem.

In certain cases it is possible to identify the solution in terms of known functions. This particular initial value problem can also be solved by the method of variation of parameters, yielding the solution:

$$f(x) = \frac{1}{2}\sin x - \frac{1}{2}x\cos x.$$

The Maclaurin series for this $f(x)$ is exactly the series we calculated.

Exercises:

1. Verify that the Maclaurin series for $f(x) = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ is exactly the power series we calculated as solution to $y'' + y = \sin x, \quad y(0) = 0, \quad y'(0) = 0$.

2. Use the power series method to solve the initial value problem

$$y' = y, \quad y(0) = 1$$

   and identify the solution as an elementary function.

3. Use the power series method to solve the initial value problem

$$y' = ky, \quad y(0) = C$$

   and identify the solution as before.

4. Use the power series method to solve the initial value problem

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

   and identify the solution as an elementary function.

5. Use the power series method to solve the initial value problem

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

   and identify the solution as an elementary function.

6. Use the power series method to solve the initial value problem

$$y'' + k^2 y = 0, \quad y(0) = A, \quad y'(0) = B$$

   and identify the solution in terms of elementary functions.

7. Use the power series method to solve the initial value problem

$$y'' + y = x, \quad y(0) = 0, \quad y'(0) = 0$$

   (consider x as a power series with $c_1 = 1$ and all other coefficients zero) and identify the solution as the power series of $f(x) = x - \sin x$.