1. Experimental approach

Problem 1.1. For the following sequences, guess an explicit formula for the general term:

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\dots$$
$$1,\ 2,\ 4,\ 7,\ 11,\ 16,\ 22,\dots$$
$$0,\ \frac{7}{2},\ 13,\ \frac{63}{2},\ 62,\ \frac{215}{2},\ 171,\dots$$

Problem 1.2. Compute the following sums

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}, \quad \sum_{k=1}^{n} k \cdot k!, \quad \sum_{k=1}^{n} (-1)^k k^2.$$

Problem 1.3. Compute the product

$$\prod_{k=1}^{n} \left(1 - \frac{1}{k^2}\right).$$

Problem 1.4. Suppose that n lines in the plane are such that no two of them are parallel and no three of them have a common point. In how many parts do these lines divide the plane?

Problem 1.5. Suppose that

$$u_1 = 2, \quad u_2 = 3, \quad u_n = 3u_{n-1} - 2u_{n-2}, \quad n > 2$$

Prove that $u_{100} - 1$ is divisible by 16.

Problem 1.6. Give an example of 10 different positive integers, whose sum is divisible by each of them.

Problem 1.7. Give an example of 5 positive integers $a_1, \dots, a_5$ such that $a_i - a_j = \gcd(a_i, a_j)$ for all $i \neq j$.

Problem 1.8. Suppose that

$$u_1 = 1, \quad u_{n+1} = u_n + 8n.$$

Prove that u_n is a square of an integer.

Problem 1.9. (Putnam 2004) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Problem 1.10. Let f_n be the n-th Fibonacci number:

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2},\ n > 2.$$

Prove that $f_n^2 + (-1)^n$ is divisible by f_{n-1}.

Problem 1.11. (Putnam 2003) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

2. Parity

Problem 2.1. Eleven cogwheels are linked in a circular order (so that the first touches the second, the second touches the third, etc., the eleventh touches the first). Can the cogwheels rotate?

Problem 2.2. Can a line not containing vertices of a (non-convex) 11-gon intersect all its edges?

Problem 2.3. Can one draw a curve on the surface of a dodecaherdon that intersects each edge once? (A dodecahedron is a regular polyhedron with 12 pentagonal faces).

Problem 2.4. Start with 4 numbers 0, 1, 0, 0. In one step, one can add one to any pair of numbers. By doing finitely many such steps, can one make all numbers equal?

Problem 2.5. Can one put plus or minus signs between all digits of the number 123456789 so that to make the result equal 0?

Problem 2.6. Seven positive integers are written on a circle. Prove that there are two adjacent numbers, whose sum is even.

Problem 2.7. Consider n integers. Prove that the sum of all pairwise distances between these integers is even (the distance between two numbers a and b is defined as $|a - b|$).

Problem 2.8. Three grasshoppers sit on the line. The left and the right grasshoppers are at distance 1 from the middle one. Any grasshopper can jump over any other grasshopper and land at the symmetric point. Suppose that, after several jumps, the grasshoppers sit at the same points, but in different order. Prove that the left one and the right one interchanged.

Problem 2.9. Can a chess knight make a series of moves, starting at cell a1, arriving at cell h8, and visiting each cell exactly once?

Problem 2.10. The numbers $x_1, \ldots, x_n$ are equal to ± 1. Suppose that $x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 = 0$. Prove that n is divisible by 4.

3. Graphs

Problem 3.1. Prove that the number of people who shook hands odd number of times is even.

There are many different formulations of this problem. To recognize the core of the problem, it is useful to reformulate it in more abstract terms. Of course, it is not very important for the problem what handshakes are. The only important thing is the following: if A shook hands with B, then B also shook hands with A the same number of times.

We can schematically represent people by points, and handshakes by arcs connecting pairs of points. Such picture is called a *graph*. The points are called the *vertices* of the graph, and the arcs connecting pairs of points are called *edges*. It is irrelevant which particular arc we choose to represent an edge, the important thing is which pairs of points are connected and which are not. Edges can intersect, and the same pair of points can be connected by several edges (as we will usually have for handshakes). The *order* of a vertex is defined as the number of edges meeting at this vertex. A graph is called *finite* if it has finitely many edges and vertices. Problem 3.1 can now be reformulated as follows:

Problem 3.2. In a finite graph, the number of vertices of odd order is even. *Hint:* count the number of pairs (v, e), where v is a vertex and e is an edge incident to v (i.e. connecting v with some other vertex).

It is important to recognize problems about graphs (which is not always easy) and to reformulate them in terms of graph theory.

Problem 3.3. In any group of 6 people, there are at least 3 pairwise acquainted or at least 3 pairwise unacquainted.

A graph is said to be *connected* if there is a path (consisting of edges) from each vertex to each. A *loop* in a graph is a nontrivial path going from a vertex to itself. A *tree* is a connected graph with no loops.

Problem 3.4. Prove that a tree with n vertices has $n-1$ edges. *Hint:* use induction on n.

Solve the following problems, first restating them in terms of graph theory:

Problem 3.5. Matches are laid on a chessboard along the boundaries of cells. How many matches do you need to remove to ensure that a rook can move from any cell to any other cell?

Problem 3.6. After 52 saw cuts, I cut my logs into 72 pieces. How many logs did I have?

Problem 3.7. A commercial shooting-range has the following rule: you pay for 5 shots, but each hit gives you 2 more bonus shots. Suppose you made 17 shots. How many times did you hit the target?

Problem 3.8. There are 100 buildings in a town. One can fence in any collection of buildings. What is the maximal number of non-intersecting fences such that no fence encloses the same collection of buildings.

Problem 3.9. Prove that any connected graph contains a tree.

Problem 3.10. In a subway system, all stations are connected (i.e. one can go from any station to any other station). Prove that it is possible to demolish one station (including the rail tracks) so that all remaining stations will still be connected.

Problem 3.11. In any group of n people, there are at least two people with the same number of acquaintances.

4. Areas

Formulas for the area of a triangle. Let ABC be a triangle with vertices A, B and C. Denote by $[ABC]$ the area of the triangle. Let a, b and c be the lengths of the sides BC, AC and AB, respectively.

(1) Altitude+side. Denote by h_a the length of the altitude at A.

$$[ABC] = \frac{1}{2}ah_a.$$

(2) Two sides+angle.

$$[ABC] = \frac{1}{2}ab \sin C.$$

(3) Three sides (Heron' formula). Denote by $p = \frac{a+b+c}{2}$ the semi-perimeter.

$$[ABC] = \sqrt{p(p-a)(p-b)(p-c)}.$$

(4) Three sides+circumradius. Denote by R the radius of the circumcircle.

$$[ABC] = \frac{abc}{4R}.$$

(5) Perimeter+inradius. Denote by r the radius of the incircle.

$$[ABC] = rp.$$

Problem 4.1. Triangle ABC has an area 1. Points D and E lie, respectively on sides BC and CA so that $DC/BC = t$ and $EC/CA = s$.
(a) Find the area of the triangle ADB.
(b) Find the area of the triangle ECD

Problem 4.2. What is the maximum possible area of a triangle with sides a and b?

Problem 4.3. For $i = 1, 2$ let T_i be a triangle with two sides of lengths a_i and b_i and the angle γ_i between these two sides. Suppose that $a_1 < a_2$, $b_1 < b_2$ and $\gamma_1 < \gamma_2$. Is it true that the area of T_1 is less than the area of T_2?

Problem 4.4. Let ABC be an equilateral triangle with the side length 1, and P a point inside the triangle. Show that the sum of distances from P to the sides of the triangle ABC does not depend on the position of P inside the triangle. Find the value of this sum.

Problem 4.5. (Putnam 2004) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i and area A_i. Suppose that $a_1 < a_2$, $b_1 < b_2$, $c_1 < c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 < A_2$?

Problem 4.6. (Putnam 2000) The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem 4.7. (Putnam 2001) Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Areas of curved regions.

(1) The area under the graph Γ of a non-negative function $f : [a, b] \to \mathbb{R}$ (and above the x-axis) is

$$\int_a^b f(x)dx = \int_\Gamma y\, dx.$$

(2) The area to the left of the graph Γ of increasing function $f : [a, b] \to \mathbb{R}$ and to the right of the y-axis is

$$\int_a^b x\, df(x) = \int_\Gamma x\, dy.$$

(3) The area of the sector given by the inequalities $r \le f(\varphi)$, $\varphi_0 \le \varphi \le \varphi_1$ in polar coordinates, is

$$\frac{1}{2}\int_{\varphi_0}^{\varphi_1} r\, d\varphi = \frac{1}{2}\int_\Gamma (x\, dy - y\, dx),$$

where Γ is the arc $r = f(\varphi)$, $\varphi_0 \le \varphi \le \varphi_1$.

(4) Consider a smooth simple closed curve

$$x = x(t), \quad y = y(t), \quad 0 \le t \le 1, \quad x(0) = x(1), \quad y(0) = y(1)$$

that bounds a region X in the plane. Assume that the point $(x(t), y(t))$ goes around X in counterclockwise direction as t runs from 0 to 1. Then the area of X is given by the formula

$$\frac{1}{2}\int_0^1 x(t)dy(t) - y(t)dx(t) = \int_{\partial X} x\, dy - y\, dx.$$

Problem 4.8. Let s be any arc of the hyperbola $xy = 1$ such that s lies entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A = B$.

Problem 4.9. Let s be any arc of the parabola $y = x^2$ such that s lies entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $2A = B$.

Problem 4.10. (Putnam 1998) Let s be any arc of the unit circle $x^2 + y^2 = 1$ such that s lies entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

5. Arithmetic modulo p

Let p be a prime number. The following facts about congruences and remainders modulo p are useful.

(1) For any integer x, the number $x^p - x$ is divisible by p (Fermat).

(2) All coefficients of the polynomial $x^p - x$ are congruent mod p to the corresponding coefficients of the polynomial

$$x(x-1)\cdots(x-p+1).$$

(3) For any integer x not divisible by p, there exists an integer y such that

$$xy \equiv 1 \pmod{p}$$

The remainder of y in division of p is uniquely defined by this property.

(4) There exists an integer a (depending on p) such that any integer is congruent to a^k mod p for some $k \in \mathbb{Z}$.

(5) $(p-1)! \equiv -1 \pmod{p}$ (Wilson).

Problem 5.1. Is it true that the number

$$n^3 + 5n - 1$$

is prime for every nonnegative integer n?

Problem 5.2. Prove that, for a prime p, the numerator m of the fraction

$$\frac{m}{n} = \sum_{k=1}^{p-1} \frac{1}{k}$$

is divisible by p. *Hint:* use (3).

Problem 5.3. Prove that, for a prime p, the numerator m of the fraction

$$\frac{m}{n} = \sum_{k=1}^{p-1} \frac{1}{k^2}$$

is divisible by p.

Problem 5.4. Prove that, for a prime p, the numerator m of the fraction

$$\frac{m}{n} = \sum_{k=1}^{p-1} \frac{1}{k}$$

is divisible by p^2.

Problem 5.5. Prove that, for any postitive integers a and b, the number

$$\binom{pa}{pb} - \binom{a}{b}$$

is divisible by p^3 for every $p \geq 3$.

Problem 5.6. Find the maximal power of 5 dividing the number

$$\binom{100}{50} - \binom{20}{10}.$$

6. The greatest common divisor and the least common multiple

The following facts are useful:

(1) Each integer n admits a unique (up to obvious transformations) prime decomposition

$$n = \prod p^{\alpha_p}$$

where p are primes and α_p are nonnegative integers. Since $\alpha_p \neq 0$ for only finitely many p, this product is finite.

(2) The gcd and lcm of two numbers are expressed as follows: if

$$a = \prod p^{\alpha_p}, \quad b = \prod p^{\beta_p},$$

then

$$\gcd(a,b) = \prod p^{\min(\alpha_p,\beta_p)}, \quad \operatorname{lcm}(a,b) = \prod p^{\max(\alpha_p,\beta_p)}.$$

(3) The gcd of two numbers is their integer linear combination: for any two integers a and b, there exist integers x and y such that $ax + by = \gcd(a,b)$.

Problem 6.1. Prove that

$$\gcd(a+b, \operatorname{lcm}(a,b)) = \gcd(a,b).$$

Problem 6.2. Are there positive integers a and b such that the fractions a/b, $(a+1)/b$, $(a+1)/(b+1)$ are simple.

Problem 6.3. Prove that $27x + 4$ and $18x + 3$ are relatively prime for every integer x.

Problem 6.4. Prove that $\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$.

Problem 6.5. Can the product of three consequtive integers be a power of an integer (i.e. a square, a cube, etc.)?

Problem 6.6. Prove that $\gcd(ab, bc, ac)$ is divisible by $\gcd(a, b, c)^2$.

Problem 6.7. (Putnam 2003) Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$.) *Hint:* use (2).

Problem 6.8. (Putnam 2000) Prove that the expression

$$\frac{gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geqslant m \geqslant 1$. *Hint:* use (3).

7. Diophantine equations

Problem 7.1. Prove that the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$$

is unsolvable in positive integers.

Problem 7.2. Find all integers x, y such that $xy = x + y$.

Problem 7.3. Find positive integers x, y and z satisfying the equation

$$x + \cfrac{1}{y + \cfrac{1}{z}} = \frac{10}{7}.$$

Problem 7.4. Find all integers x, y such that $3x + 5y = 1$.

Problem 7.5. Prove that there are infinitely many triples of integers $x > 1$, $y > 1$ and $z > 1$ such that $x! = y! \cdot z!$

Problem 7.6. Is there a sphere in $\mathbb{R}^3$, which has only one rational point (a point is said to be *rational* if it has rational coordinates)?

Problem 7.7. Prove that there are infinitely many integer solutions of the equation $x^2 + y^3 = z^5$.

Problem 7.8. Find all integer solutions of the equation $x^2 + y^2 + z^2 = 2xyz$.

Problem 7.9. Prove that the following equations

(1) $x^2 + 1 = py$,
(2) $x^2 + x + 1 = py$

are solvable in integers for infinitely many primes p.

8. Inequalities

Problem 8.1. For all positive real values of x and y, prove the following inequalities:

$$\frac{x^2+y^2}{2} \geq xy, \qquad x+\frac{1}{x} \geq 2, \qquad \sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}.$$

Problem 8.2. Prove the following inequalities for positive values of all variables:

(1) $x^2+y^2+z^2 \geq xy+yz+xz$,
(2) $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$.

Problem 8.3. Prove that if $a+b+c=0$, then $ab+bc+ac \leq 0$.

Problem 8.4. For any collection of positive numbers $a_1,\dots,a_n$ and $b_1,\dots,b_n$,

$$\min_k \frac{a_k}{b_k} \leq \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \max_k \frac{a_k}{b_k}$$

Problem 8.5. Let $\varphi : [a,b] \to \mathbb{R}$ be a twice differentiable function such that $\varphi'' \geq 0$ on $[a,b]$. Prove that the graph of φ lies below the straight line connecting the points $(a,\varphi(a))$ with $(b,\varphi(b))$. Deduce that

$$\varphi(\lambda a+(1-\lambda)b) \leq \lambda\varphi(a)+(1-\lambda)\varphi(b)$$

for any $\lambda \in [0,1]$.

Jensen's inequality is the following statement: suppose that a smooth function $\varphi : [a,b] \to \mathbb{R}$ satisfies the inequality $\varphi'' \geq 0$ on $[a,b]$. Then, for any number of points $a_1,\dots,a_n \in [a,b]$ and any non-negative coefficients $\lambda_1,\dots,\lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\varphi\left(\sum_{k=1}^n \lambda_k a_k\right) \leq \sum_{k=1}^n \lambda_k \varphi(a_k).$$

Problem 8.6. Prove Jensen's inequality.

Problem 8.7. (The AM-GM inequality) For positive numbers $a_1,\dots,a_n$, prove that

$$(a_1\dots a_n)^{1/n} \leq \frac{a_1+\cdots+a_n}{n}.$$

Problem 8.8. Suppose that $a \geqslant 0$, $b \geqslant 0$ and $p,q \geqslant 1$ are s.t. $1/p+1/q=1$. Then $ab \leqslant a^p/p+b^q/q$.

Problem 8.9. **The Hölder inequality I.** Prove that, for positive values of all variables,

$$a_1b_1+\cdots+a_nb_n \leq \left(a_1^p+\cdots+a_n^p\right)^{1/p}\left(b_1^q+\cdots+b_n^q\right)^{1/q},$$

where p and q are as in Problem **??**.

Problem 8.10. **The Hölder inequality II.** For a continuous function $f : [0,1] \to \mathbb{R}$, set

$$||f||_p = \int_0^1 |f(x)|^p \, dx.$$

Prove that

$$|\int_0^1 fg d\mu| \leqslant ||f||_p ||g||_q,$$

where $1/p + 1/q = 1$.

Problem 8.11. Prove that

$$\left| \int_0^1 |f|^{p-1} g \right| \leq ||f||_p^{p-1} \, ||g||_p.$$

Problem 8.12. **The Minkowski inequality.** Show that

$$||f+g||_p \leq ||f||_p + ||g||_p$$

Hint: we need to prove that

$$||f+\lambda g||_p \leq ||f||_p + \lambda ||g||_p$$

for all $\lambda \geq 0$. Compare the λ-derivatives of the both sides.

Problem 8.13. (Putnam 2003) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ \leqslant [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

9. Polynomials

Problem 9.1. Let a and b be integers. If the number a^2+b^2 is divisible by ab, then $a = b$.

Any symmetric polynomial in x and y can be expressed as a polynomial in $u = xy$ and $v = x+y$. The same with rational functions.

Problem 9.2. Let x and y be the two roots of the quadratic equation

$$t^2 - 3t + 1 = 0.$$

Find

$$(a) \quad x^5 + y^5, \quad (b) \quad \frac{1}{x^3} + \frac{1}{y^3}, \quad (c) \quad (x+1)y^2 + x^2(y+1).$$

Problem 9.3. Express the following functions through $\sin\varphi + \cos\varphi$:

$$(a) \quad \sin\varphi\cos\varphi, \quad (b) \quad \sin^3\varphi + \cos^3\varphi,$$

$$(c) \quad \sin\varphi + \cos\varphi + \tan\varphi + \cot\varphi + \sec\varphi + \csc\varphi.$$

Problem 9.4. (Putnam 2003) Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Problem 9.5. Let λ be the biggest root of the quadratic equation

$$x^2 - 7x + 1 = 0.$$

Find the 10th decimal digit of λ^{2005}.

Problem 9.6. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with real coefficients, all of whose complex roots are on the unit circle. Prove that $a_k = a_{n-k}$ for all $k = 0, \dots, n$.

Problem 9.7. (Putnam 2005) Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z)$ have absolute value 1.

10. Probability

Suppose that there are n (disjoint) possible outcomes of a certain random experiment having probabilities $p_1, p_2, \dots, p_n$. In many problems, it is assumed that all probabilities p_i are equal (uniform probability), then $p_i = 1/n$. The probability of having any outcome from a set $S \subset \{1, \dots, n\}$ is equal to

$$\sum_{i \in S} p_i.$$

If two random experiments are *independent*, then the probability of having a certain pair of outcomes (i, j) is $p_i p_j$.

Problem 10.1. There are 8 cards with numbers 2, 4, 6, 7, 8, 11, 12 and 13 written on them. We pick two cards at random. What is the probability that the corresponding numbers are relatively prime? *Hint:* the probability of each pair is

$$\frac{2}{8(8-1)} = \frac{1}{28}$$

Problem 10.2. A box contains $2n$ white balls and $2n$ black balls. We pick $2n$ balls at random. What is the probability that we pick the same number of white and black balls?

Problem 10.3. A box contains white and black balls. The ratio of the number of white balls to the number of black balls is α. We pick balls randomly from the box until the box is empty. What is the probability that the last ball is black?

Problem 10.4. Twenty people are seated randomly on a bench (or at a round table). In each case, find the probability that a certain pair of people sit next to each other.

Problem 10.5. (Putnam 2002) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Geometric probabilities. Suppose that x and y are independent random variables uniformly distributed on the segment $[0, 1]$. Let $X \subset [0, 1]^2$ be a subset of the unit square. Then the probability that $(x, y) \in X$ is equal to the area of X.

Problem 10.6. Two people agreed to meet between 10am and 11am on a bridge. Suppose that they come to the bridge at a random time between 10am and 11am, and wait for 10 minutes only.

(a) What is the probability that they meet?
(b) The same problem with 3 people?
(c) What is the probability that at least 2 out of 3 people meet?

Problem 10.7. What is the probability that the quadratic equation $x^2 + px + q = 0$ will have real roots if p and q are chosen randomly and independently from the segment $[-1, 1]$.

Problem 10.8. We throw randomly 3 points on a segment (one point at a time). What is the probability that the third point falls between the first two?

Problem 10.9. (Putnam 1993) Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + \pi s$, where r and s are rational numbers.

Problem 10.10. (Putnam 1992) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)