

## Axiom of Choice

The *Axiom of Choice* states that for any collection  $\mathcal{S}$  of sets there exists a *choice function*  $\sigma: \mathcal{S} \rightarrow \bigcup \mathcal{S}$  such that  $\sigma(X) \in X$  for all  $X \in \mathcal{S}$ . We will derive the Zorn lemma directly from the axiom of choice.

**Theorem 1 (Zorn)** *Let  $X$  be a partially ordered set. Suppose that for any linearly ordered subset  $A$  of  $X$  there is an upper bound, i.e. an element  $\hat{a} \in X$  such that  $a \leq \hat{a}$  for all  $a \in A$ . Then  $X$  has a maximal element, i.e. an element  $\hat{x} \in X$  such that the inequality  $x \geq \hat{x}$  implies  $x = \hat{x}$ .*

PROOF. The proof goes in several steps.

1) Fix any element  $x_0 \in X$ .

2) Suppose that  $X$  has no maximal elements. Then for any linearly ordered subset  $A \subset X$ , the set of all upper bounds not belonging to  $A$ , is nonempty. Choose one upper bound  $\sigma(A)$  from this set. Thus we have  $A \leq \sigma(A)$  (i.e.,  $a \leq \sigma(A)$  for all  $a \in A$ ) and  $a \notin A$ . Let us fix this choice function  $\sigma$  for the rest of the proof.

3) Define a *chain* in  $X$  to be a subset  $C \subset X$  satisfying the following properties:

- We have  $x_0 \in C$ .
- The subset  $C$  is well-ordered.
- For any  $x \in C$ , we have  $\sigma\{y \in C \mid y < x\} = x$

4) Chains exist, e.g.  $\{x_0\}$  is a chain.

5) If  $C$  is a chain and  $x \in C$ , then  $C_x = \{y \in C \mid y < x\}$  is also a chain (this is obvious). By definition of chains, we have  $\sigma(C_x) = x$ .

6) Suppose that  $C' \subset C$  are chains. If  $C' \neq C$ , then there is an element  $x \in C$  such that  $C' = C_x$ . To prove this, we first define  $x$  to be the minimal (smallest) element of the set  $C - C'$ . Then it is clear that  $C_x \subset C'$ . Let  $x'$  be the minimal element of  $C' - C_x$ , so that we have  $C'_{x'} \subset C_x$ . The element  $x'$  cannot be less than  $x$ , otherwise it would be in  $C_x$ . Hence,  $x' \geq x$ . It follows that  $C'_{x'} \supset C_x$ . We see that  $C_x = C'_{x'}$  as we have proved both inclusions. But now  $\sigma(C_x) = x$  and  $\sigma(C'_{x'}) = x'$ , therefore,  $x' = x$ . But  $x'$  belongs to  $C'$  whereas  $x$  does not. Contradiction.

7) For any pair of chains  $C$  and  $C'$ , we have either  $C \subset C'$  or  $C' \subset C$ . Indeed, consider the chain  $C \cap C'$ . By step 6), if it does not coincide with either  $C$  or  $C'$ , then it has the form  $C_x = C'_{x'}$  for some  $x \in C$  and  $x' \in C'$ . In particular, the elements  $x$  and  $x'$  do not belong to the intersection  $C \cap C'$ , while  $x$  belongs to  $C$  and  $x'$  belongs to  $C'$ . But now  $x = \sigma(C_x) = \sigma(C'_{x'}) = x'$ . Contradiction.

8) Take the union of  $\hat{C}$  all chains. We claim that  $\hat{C}$  is also a chain. It is clear from steps 6) and 7) that  $\hat{C}$  is linearly ordered and that  $\sigma(\hat{C}_x) = x$  for all  $x \in \hat{C}$ . It only remains to prove that  $\hat{C}$  is well-ordered. [This is an exercise from Homework 4].

9) The chain  $\hat{C}$  is the largest chain. Now consider  $\hat{x} = \sigma(\hat{C})$ . Clearly,  $\hat{C} \cup \{\hat{x}\}$  is a chain that is bigger than  $\hat{C}$ . Contradiction, which proves the Zorn lemma.  $\square$

**Theorem 2 (Zermelo)** *Any set admits a well-ordering.*

PROOF. This theorem can be proved in almost the same way as the Zorn lemma. Namely, we need to fix a point  $x_0 \in X$ , and a choice function  $\sigma$  that maps any subset  $A \subset X$  to a point  $\sigma(A) \notin A$ .

Define a *chain* in  $X$  to be a subset  $C \subset X$  together with a well-ordering of  $C$  such that  $x_0$  is the minimal element of  $C$ , and  $\sigma\{y \in C \mid y < x\} = x$  for any  $x \in C$ .

We can repeat steps 4)-9) to show that the union  $\hat{C}$  of all chains is well-ordered. The chain  $\hat{C}$  is the largest, i.e. it contains all other chains. To get the contradiction, we now can define an order on  $\hat{C} \cup \{\sigma(\hat{C})\}$  such that this will be a chain. It suffices to declare  $\sigma(\hat{C})$  to be the largest element. Contradiction with the fact that  $\hat{C}$  is the largest chain.  $\square$