Math Battle II

Problem 1. Let x_1 and x_2 be the roots of a quadratic equation $ax^2 + bx + c = 0$. Set $S_n = x_1^n + x_2^n$. Prove the following formula:

$$aS_n + bS_{n-1} + cS_{n-2} = 0, \quad m \geq 2$$

Problem 2. In how many ways can one triangulate (i.e. split into triangles) a convex n-gon without adding new vertices?

Problem 3. Compute the following integral:

$$\int_0^{\frac{\pi}{2}} (\cos^2(\cos x) + \sin^2(\sin x))\, dx.$$

Problem 4. Compute the following sum:

$$\sum_{k=0}^{[n/3]} \binom{n}{3k}.$$

Here $[n/3]$ denotes the maximal integer not exceeding $n/3$.

Problem 5. Consider a random permutation σ on n elements (all permutations on n elements are assumed to have the same probability). Let P_n denote the probability that $\sigma(x) = x$ for at least one x. Find the limit $\lim_{n\to\infty} P_n$.

Problem 6. Consider a parallelogram with vertices in $\mathbb{Z}^2$ (where $\mathbb{Z}^2$ denotes the set of points (x, y) such that both x and y are integer). Suppose that it only intersects $\mathbb{Z}^2$ at the vertices. Prove that the area of the parallelogram is 1.