Math Battle I

Problem 1. Let E and V denote the number of edges and vertices, respectively, in a convex bounded 3-dimensional polyhedron. Prove that $E \leq 3V - 6$.

Hint/Solution. Let E_i be the number of edges on the i-th face. We have $\sum E_i = 2E$, because each edge is incident to exactly 2 faces. On the other hand, $E_i \geq 3$. It follows that $3F \leq 2F$. Combined with the Euler theorem $V - E + F = 2$, this yields the result.

Problem 2. Consider three pair-wise intersecting circles. For each pair of circles, draw the line through their intersection point. Prove that these three lines are either all parallel or all pass through the same point.

Hint/Solution. Let the equation of the i-th circle be $x^2 + y^2 = l_i(x, y)$, where l_i is a linear function. The intersection of the i-th and the j-th circle lies on the line $l_i = l_j$. If the lines $l_1 = l_2$ and $l_2 = l_3$ intersect, then the intersection point is given by the equations $l_1 = l_2 = l_3$, and this point also lies on the line $l_1 = l_3$.

Problem 3. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = e^{-x}$ for all $x \in \mathbb{R}$?

Hint/Solution. The function f must be injective, since $f \circ f$ is injective. Any injective continuous function is strictly monotone. The composition of a strictly monotone function with itself is strictly increasing. Thus the answer is negative.

Problem 4. Is there a sequence of positive integers such that any positive integer can be uniquely represented as a difference of two terms in the sequence?

Hint/Solution. Yes. Let us construcut this sequence $a_1, a_2, \ldots, a_n, \ldots$ inductively. Suppose that the finite sequence $a_1, \ldots, a_n$ represents all numbers up to k, but does not represent $k + 1$. Then set $a_{n+1} = 2a_n + 1$ and $a_{n+2} = a_{n+1} + k + 1$.

Problem 5. Positive numbers a, b and c are such that $abc = 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+a+c} \leq 1$$

Hint/Solution. Set $a = x^3$, $b = y^3$ and $c = z^3$. Prove that

$$\frac{1}{1+a+b} \leq \frac{z}{x+y+z}.$$

Problem 6. Does the sequence $1+17n^2$, $n \in \mathbb{N}$, contain an infinite number of squares?

Hint/Solution. Suppose that $m^2 - 17n^2 = 1$. We have $(m + \sqrt{17}n)^2 = a + \sqrt{17}b$ for some integers a and b. Prove that $a^2 - 17b^2 = 1$.