Inequalities

Problem 1. For all positive real values of x and y, prove the following inequalities:

$$\frac{x^2+y^2}{2} \geq xy, \qquad x+\frac{1}{x} \geq 2, \qquad \sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}.$$

Problem 2. Prove the following inequalities for positive values of all variables:

(1) $x^2+y^2+z^2 \geq xy+yz+xz$,
(2) $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$.

Problem 3. Prove that if $a+b+c=0$, then $ab+bc+ac \leq 0$.

Problem 4. For any collection of positive numbers $a_1,\dots,a_n$ and $b_1,\dots,b_n$,

$$\min_k \frac{a_k}{b_k} \leq \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \max_k \frac{a_k}{b_k}$$

Problem 5. Let $\phi : [a,b] \to \mathbb{R}$ be a twice differentiable function such that $\phi'' \geq 0$ on $[a,b]$. Prove that the graph of ϕ lies below the straight line connecting the points $(a,\phi(a))$ with $(b,\phi(b))$. Deduce that

$$\phi(\lambda a+(1-\lambda)b) \leq \lambda\phi(a)+(1-\lambda)\phi(b)$$

for any $\lambda \in [0,1]$.

Jensen's inequality is the following statement: suppose that a twice differentiable function $\phi : [a,b] \to \mathbb{R}$ satisfies the inequality $\phi'' \geq 0$ on $[a,b]$. Then, for any number of points $a_1,\dots,a_n \in [a,b]$ and any non-negative coefficients $\lambda_1,\dots,\lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\phi\left(\sum_{k=1}^n \lambda_k a_k\right) \leq \sum_{k=1}^n \lambda_k \phi(a_k).$$

Problem 6. Prove Jensen's inequality.

Problem 7. (The AM-GM inequality) For positive numbers $a_1,\dots,a_n$, prove that

$$(a_1 \dots a_n)^{1/n} \leq \frac{a_1+\cdots+a_n}{n}.$$

Problem 8. Suppose that $a \geq 0$, $b \geq 0$ and $p,q \geq 1$ are s.t. $1/p+1/q=1$. Then $ab \leq a^p/p+b^q/q$.

Problem 9. Prove that, for positive values of all variables,

$$a_1b_1+\cdots+a_nb_n \leq \left(a_1^p+\cdots+a_n^p\right)^{1/p}\left(b_1^q+\cdots+b_n^q\right)^{1/q},$$

where p and q are as in Problem 8.