Graphs

Problem 1. Prove that the number of people who shook hands odd number of times is even.

There are many different formulations of this problem. To recognize the core of the problem, it is useful to reformulate it in more abstract terms. Of course, it is not very important for the problem what handshakes are. The only important thing is the following: if A shook hands with B, then B also shook hands with A the same number of times.

We can schematically represent people by points, and handshakes by arcs connecting pairs of points. Such picture is called a *graph*. The points are called the *vertices* of the graph, and the arcs connecting pairs of points are called *edges*. It is irrelevant which particular arc we choose to represent an edge, the important thing is which pairs of points are connected and which are not. Edges can intersect, and the same pair of points can be connected by several edges (as we will usually have for handshakes). The *order* of a vertex is defined as the number of edges meeting at this vertex. A graph is called *finite* if it has finitely many edges and vertices. Problem 1 can now be reformulated as follows:

Problem 2. In a finite graph, the number of vertices of odd order is even. *Hint:* count the number of pairs (v, e), where v is a vertex and e is an edge incident to v (i.e. connecting v with some other vertex).

It is important to recognize problems about graphs (which is not always easy) and to reformulate them in terms of graph theory.

Problem 3. In any group of 6 people, there are at least 3 pairwise acquainted or at least 3 pairwise unacquainted.

A graph is said to be *connected* if there is a path (consisting of edges) from each vertex to each. A *loop* in a graph is a nontrivial path going from a vertex to itself. A *tree* is a connected graph with no loops.

Problem 4. Prove that a tree with n vertices has $n-1$ edges. *Hint:* use induction on n.

Solve the following problems, first restating them in terms of graph theory:

Problem 5. Matches are laid on a chessboard along the boundaries of cells. How many matches do you need to remove to ensure that a rook can move from any cell to any other cell?

Problem 6. After 52 saw cuts, I cut my logs into 72 pieces. How many logs did I have?

Problem 7. A commercial shooting-range has the following rule: you pay for 5 shots, but each hit gives you 2 more bonus shots. Suppose you made 17 shots. How many times did you hit the target?

Problem 8. There are 100 buildings in a town. One can fence in any collection of buildings. What is the maximal number of non-intersecting fences such that no fence encloses the same collection of buildings.

Problem 9. Prove that any connected graph contains a tree.

Problem 10. In a subway system, all stations are connected (i.e. one can go from any station to any other station). Prove that it is possible to demolish one station (including the rail tracks) so that all remaining stations will still be connected.

Problem 11. In any group of n people, there are at least two people with the same number of acquaintances.