Functional equations

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that $f(mx/n) = mf(x)/n$ for any integer m, any non-zero integer n, and any $x \in \mathbb{R}$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Prove that there is a number $a \in \mathbb{R}$ such that $f(x) = ax$ for all x.

There is a discontinuous function f with the property $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. A proof of this fact is not very hard, but it requires some technique from set theory (e.g. the Zorn lemma).

Problem 3. The graph of such function is everywhere dense in the plane, i.e. for every point (x, y) and any $\varepsilon > 0$, there is a number x' such that the distance from (x, y) to the point $(x', f(x'))$ on the graph of f is less than ε.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) \neq 0$ for at least one $x \in \mathbb{R}$ (such functions are called *field automorphisms* of $\mathbb{R}$).

Problem 5. Find all continuous solutions f to the following functional equations:

(1) $f(x+y) = f(x)f(y)\ \forall x, y \in \mathbb{R}$,
(2) $f(xy) = f(x)f(y)\ \forall x, y \in \mathbb{R}$,
(3) $f(x+y) + f(x-y) = 2f(x)f(y)\ \forall x, y \in \mathbb{R}$.

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following functional equation

$$f(x) + (x + \frac{1}{2})f(1-x) = 1, \quad \forall x \in \mathbb{R}.$$

Problem 7. Is there a function $f : \mathbb{R} \to \mathbb{R}$ (not necessarily continuous) such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$?