Combinatorics

Problem 1. Each tenth mathematician is a philosopher. Each hundredth philosopher is a mathematician. Are there more mathematicians than philosophers or more philosophers than mathematicians?

Problem 2. An ogre has 25 captives.

(1) In how many ways can he choose 3 captives for breakfast, lunch and dinner?
(2) In how many ways can he choose 3 captives to let them go?

Problem 3. Which number is bigger: the number of six-digit integers representable as a product of two three-digit integers, or the number of six-digit integers not representable in this form?

Problem 4. Suppose that we have n teacups and m teaspoons. In how many ways one can put the teaspoons into the teacups? All teacups and teaspoons are distinguishable from each other.

It is probably not so easy to see that the following problem is the same as the first one:

Problem 5. A die with n facets is rolled m times in a row. How many different outcomes are possible?

These problems can be restated in set-theoretic language as follows:

Problem 6. How many maps are their from a set with m elements to a set with n elements?

Answer: n^m.

Before solving the following problems, reformulate them in set-theoretic language:

Problem 7. The same question as in Problem 4, but now in each cup, there can be at most one spoon.

Answer: $n!/(n-m)!$.

Problem 8. The same question as in Problem 4, but now each cup must have at least one spoon.

Here the answer is less satisfactory. It can be obtained by the *inclusion-exclusion principle*:

$$F(n,m) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m.$$

Problem 9. Prove that the number $F(n,m)$ is divisible by $n!$.

The following is a simpler exercise on the inclusion-exclusion principle:

Problem 10. Find the number of all positive integers up to 1000 that are divisible neither by 2, nor by 3, nor by 5.

Problem 11. The same question as in Problem 4, but now the spoons are indistinguishable (however, the cups are still distinguishable). In other words, we can only say how many spoons are in each particular cup, but we cannot say which spoons.

Answer: $\binom{n+m-1}{n-1}$.

Problem 12. The same question as in Problem 4, but now the cups are indistinguishable (however, the spoons are distinguishable). In other words, we can only say which collections of spoons are in the same cups, but we cannot say in which cups.

The answer to this problem is even less satisfactory than that to Problem 8.

Answer:

$$\sum_{k=1}^{n} \frac{F(k,m)}{k!}$$

Problem 13. The same question as in Problem 4, but now both cups and spoons are indistinguishable.

The answer to this problem is the most unsatisfactory, if it can be called an answer at all.

Answer: $G(n,m)$ is the coefficient with x^m in the power expansion of the following function:

$$\frac{1}{(1-x)(1-x^2)\cdots}$$