GAUSSIAN CURVATURE DECAY FOR CONFORMALLY FLAT METRICS ON THE PLANE

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Abstract. The decay rate of the Gauss curvature of conformally flat planer surfaces of strictly negative curvature is studied. It is shown that generically there is an asymptotic sequence that decays faster than quadratically in the distance from the origin. In the case that the conformal factor is of finite order, it is shown that one can improve this decay rate.

1. introduction

There have been many investigations of the properties of surfaces with negative Gaussian curvature. One of the principal results in this area is due to Efimov [5] which states that no surface S with Gaussian curvature $K \leq \delta < 0$ for any $\delta < 0$ can be C^2 immersed into $\mathbb{R}^3$ so that S is complete in the induced Riemannian metric. Establishing the optimal C^2 regularity was a difficult problem (see [6] for a brief history). Thus Efimov's theorem requires that any complete C^2 immersed surface $S \subset \mathbb{R}^3$ with negative Gaussian curvature $K \leq 0$ must have $K \to 0$ along some sequence of points in S. Our main aim is to determine the rate at which such a sequence vanishes under certain assumptions on the conformal factor of a C^2 metric on the surface without an immersion hypothesis.

Let $g = e^u|dz|^2$ be a conformally flat metric on $\mathbb{C}$ where $\{z, \bar{z}\}$ are the standard local coordinates on the plane. We will often identify $\mathbb{C}$ with $\mathbb{R}^2$ via the canonical diffeomorphism. The Gaussian curvature of g is given by $K = -e^{-u}\Delta u/2 = -\Delta_g u/2$, which can be represented by the equation

$$\Delta u + 2Ke^u = 0. \tag{1.1}$$

Here we used the notation $\Delta_g = g^{ij}\partial_i\partial_j = 4e^{-u}\partial_z\partial_{\bar{z}}$ for the metric Laplacian and Δ denotes the Euclidean Laplacian. In particular, the solution space of this equation for a prescribed K represents all admissible functions that can be used to define conformally flat metrics on the plane with Gaussian curvature K. For instance, if we take $K = 0$, then any harmonic function u defines a metric g of trivial Gaussian curvature. Throughout this paper, we will suppose that $K < 0$, which by the previous equation implies that $u : \mathbb{C} \to \mathbb{R}$ is a strictly subharmonic function. We will also use the notation $|z|$ for the modulus of z with respect to the Euclidean metric. We note that the local coordinates z and $\bar{z}$ are in fact global since they cover the plane in one chart .

Frequent use will be made of the following definition: If u is a subharmonic function on $\mathbb{C}$ (but not harmonic), then the limit

$$\beta = \lim_{r\to\infty} \frac{I(r,u)}{\ln r} \quad \text{where} \quad I(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})dt, \tag{1.2}$$

exists and $\beta \in (0, \infty]$ (cf. section 2), and we say that u is of order β.

We first analyze the asymptotic behavior of K under the constraint that the function u in the conformal factor of g is of finite order β. In section 2 we prove

Theorem 1. *Let $e^u|dz|^2$ be a conformally flat metric on $\mathbb{C}$ with Gaussian curvature $K<0$, and suppose u is of finite order β. Then there is no triple (s, r_0, C) with $s<\beta+2$, $C>0$, and $r_0>0$ such that*

$$K(z) \leq -\frac{C}{2|z|^s} \quad \text{for all} \quad |z|>r_0. \tag{1.3}$$

In particular, there exists a sequence $\{z_n\}$ with $|z_n|\to\infty$ such that $|z_n|^{\beta^*+2}K(z_n)\to 0$ for any $\beta^*\in(0,\beta)$.

We later give an example to show that this theorem is false if instead one assumes $s>\beta+2$. It would be interesting to find a counterexample or proof for the case $s=\beta+2$. The proof of this theorem requires β to be finite. We next turn to the case where $\beta=\infty$ and find a weaker result,

Theorem 2. *Let $e^u|dz|^2$ be a conformally flat metric on $\mathbb{C}$ with Gaussian curvature $K<0$, and suppose u is of order $\beta=\infty$. Then there is no pair (r_0, C) with $C>0$ and $r_0>0$ such that*

$$K(z) \leq -\frac{C}{2|z|^2} \quad \text{for all} \quad |z|>r_0. \tag{1.4}$$

We can combine these theorems to get a statement that is independent of β,

Corollary 1. *Let $e^u|dz|^2$ be a conformally flat metric on $\mathbb{C}$ with Gaussian curvature $K<0$. Then there does not exist a $C>0$ and an $r_0>0$ such that*

$$K(z) \leq -\frac{C}{2|z|^2} \quad \text{for all} \quad |z|>r_0. \tag{1.5}$$

In particular there exists a sequence $\{z_n\}$ with $|z_n|\to\infty$, such that $|z_n|^2K(z_n)\to 0$.

Note that the hypothesis that the metric is conforamally flat on the plane is crucial. Without this assumption, the following example demonstrates that the theorem is false:

Let ρ be the Poincaré metric on the unit disk and consider a diffeomorphism $\omega:\mathbb{R}^2\to D$ where D is the unit disk, given by $\omega(z)=\frac{z}{1+|z|}$. Choose a metric $\tilde{\rho}$ on $\mathbb{R}^2$ such that ω is an isometry between (D,ρ) and $(\mathbb{R}^2,\tilde{\rho})$. Then $K_{\tilde{\rho}}\equiv -1$, and the sequence conclution of the corollary is false.

A version of Corollary 1 is proved in [7] using different techniques. In [7], the cases where u is of finite or infinite order are not distinguished. If u is of finite order, Theorem 1 requires the curvature to decay faster than in the generic case along some sequence.

Finally, we note that Corollary 1 can be rephrased to show non existence of solutions of the Liouville equation in the plane under certain conditions.

Corollary 2. *Let $K(z)\leq -C/|z|^2$ on $\mathbb{R}^2-B(0,R)$ for some $R,C>0$. Then there is no solution u of the Liouville equation (1.1) on the plane.*

There are also statements analogous to this corollary for both theorems.

2. Proof of Theorem 1

Let $(\mathbb{C}, e^u|dz|^2)$ be a Riemannian surface with negative Gaussian curvature, and let $|z|$ be the standard norm on $\mathbb{C}$. Define $A(s,r) = \{z \in \mathbb{C} | s \le |z| \le r\}$ to be the annulus bounded by the balls $B_s(0)$, $B_r(0)$ where $\partial B_s(0)$ is given the clockwise orientation and $\partial B_r(0)$ the opposite orientation. Then for $0 \le r_0 < r$, we use the divergence theorem to compute

(2.1)
$$\frac{1}{2\pi}\int_{A(r_0,r)} (\Delta_g u) d\mu = \frac{1}{2\pi}\int_{A(r_0,r)} (4e^{-u}\partial_z\partial_{\bar{z}}u)e^u dxdy = \frac{1}{2\pi}\int_{A(r_0,r)} \Delta[u(\rho,\theta)]\,\rho d\rho d\theta$$

$$= \frac{1}{2\pi}\int_{\partial A(r_0,r)} (\partial_r u)\, r d\theta = rI'(r,u) - r_0 I'(r_0,u), \tag{2.2}$$

where $I(r,u)$ is the integral mean of u defined in the introduction, ∂_r is the outward unit (w.r.t. the Euclidean metric) normal vector field to $A(r_0,r)$, $d\mu = i/2\sqrt{\det(g)}\, dz \wedge d\bar{z}$ is the volume form on $(\mathbb{C}, g)$, and $\{\rho, \theta\}$ are polar coordinates on $\mathbb{R}^2$. Now from equation (1.1) note that $\Delta u > 0$ since we suppose $K < 0$. Therefore

$$rI'(r,u) - r_0 I'(r_0,u) > 0, \tag{2.3}$$

for any r_0. In particular, letting $r_0 \to 0$ we see that $rI'(r,u) > 0$. Therefore

$$\beta = \lim_{r\to\infty} \frac{I(r,u)}{\ln r} = \lim_{r\to\infty} rI'(r) \in (0,\infty], \tag{2.4}$$

For the first theorem we restrict to the case where β is finite, i.e. u is of finite order.

Suppose $e^u|dz|^2$ is a conformally flat metric with negative Gaussian curvature K and, by way of contradiction, suppose that for a fixed $r_0 > 0$ there is an $s < \beta + 2$ and $C > 0$ such that,

$$K(z) \le -\frac{C}{2|z|^s}, \quad \text{for} \quad |z| > r_0,\ C > 0. \tag{2.5}$$

Since $r_0 I'(r_0) > 0$, for $r_0 < r/2$ we use equation (2.2) to compute

$$\begin{aligned} rI'(r) &> -\frac{1}{\pi}\int_{A(r/2,r)} K(\rho,\theta)e^u \rho d\rho d\theta = -\frac{1}{\pi}\int_{r/2}^{r}\int_0^{2\pi} K(\rho e^{i\theta})e^{u(\rho e^{i\theta})}\rho d\rho d\theta \\ &> -\frac{C}{2\pi}\int_{r/2}^{r}\int_0^{2\pi} \rho^{1-s}e^{u(\rho e^{i\theta})} d\rho d\theta > \frac{C}{2\pi}\left(r - \frac{r}{2}\right)\int_0^{2\pi}\left(\frac{r}{2}\right)^{1-s} e^{u(re^{i\theta}/2)}d\theta \\ &= \frac{\tilde{C}}{r^{s-2}}\frac{1}{2\pi}\left[\int_0^{2\pi} e^{u(re^{i\theta})/2}d\theta\right] \ge \frac{\tilde{C}}{r^{s-2}}\exp\left(\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}/2)d\theta\right) \end{aligned}$$

where we have used Jensen's inequality to establish the last inequality and introduced a new constant $\tilde{C}$. Hence we find

$$rI'(r) \ge \frac{\tilde{C}}{r^{s-2}} e^{I(r/2,u)}. \tag{2.6}$$

We wish to take a limit of this inequality. First we note that since u is of finite order β, asymptotically $I(r/2,u) \approx \beta \ln r$ and hence $e^{I(r/2,u)} \approx r^\beta$. Thus

$$\beta = \lim_{r\to\infty} rI'(r) \ge C \lim_{r\to\infty} r^{2-s+\beta}, \tag{2.7}$$

so if $s < \beta + 2$, β is not finite and we have a contradiction, which establishes Theorem 1.

3. Proof of Theorem 2

Our method of proof for the first theorem does not generalize to the $\beta = \infty$ case. We will thus proceed with a slightly different strategy. Again, by way of contradiction, suppose the hypotheses of Theorem 2 hold and that $K \le -C/|z|^2$ for $C > 0$ and all z that satisfy $|z| > r_0$ for some sufficiently large r_0. We will use the fact that $\Upsilon = e^{I(r,u)}$ is a convex function to establish inequalities that contradict this curvature bound. With this in mind, we first prove a lemma:

Lemma 1. *Let $\alpha > 1$, $s > 0$, and $g \in C^1(x_0, \infty)$ with $x_0 > 0$ where $g > 0$, $g' > 0$ and $\lim_{x\to\infty} g(x) = \infty$. Define $E_s^\alpha = \{x > x_0 : xg'(x) \ge sg^\alpha(x)\}$. Then $\mu(E_s^\alpha) < \infty$ where μ is the measure $\mu(A) = \int_A x^{-1}dx$ with $A \subset (x_0, \infty)$.*

Proof: Using the fact that $E_s^\alpha \subset (x_0, \infty)$ we compute

$$\frac{1}{\alpha - 1}\frac{1}{g^{\alpha-1}} = \int_{x_0}^\infty \frac{g'(x)}{g^\alpha(x)}dx \ge \int_{E_s^\alpha} \frac{g'(x)}{g^\alpha(x)}dx \ \ge s\int_{E_s^\alpha} \frac{dx}{x} = s\mu(E_s^\alpha) \tag{3.1}$$

which proves the lemma. We will only need the following corollary:

Corollary 3. *Let $\alpha > 1$, $s > 0$, and $x_0 > 0$. If g, g_1 satisfy the conditions of Lemma 1, then for each $x_1 > x_0$ there exists an $x > x_1$ such that $g'(x) < sg^\alpha(x)$ and $g_1'(x) < sg_1^\alpha(x)$.*

We now proceed with the proof of the theorem. Again let $e^u|dz|^2$ be a conformally flat metric where now u has infinite order. Suppose $K(z) \le -C/2|z|^2$ and note from the proof of the previous theorem that $rI'(r,u) \ge \tilde{C}\exp(I(r/2,u))$. Let $\gamma > 1$. Since $I(u,r) \to \infty$ for large r, we have

$$I'(r) \ge \frac{\tilde{C}}{r}e^{I(r/2,u)} \ge \frac{\gamma}{r}. \tag{3.2}$$

Integrating the inequality, we see $I(r,u) \ge \gamma \ln r + C$, from which we find $\exp(I(r/2,u)) \ge \tilde{C}r^\gamma$. Thus $rI'(r,u) \ge \tilde{C}e^{I(r/2)} \ge \tilde{C}r^\gamma$ which shows $I'(u,r) \to \infty$.

Now from equation (2.2), we compute

$$(rI'(r))' = -\frac{r}{\pi}\int_0^{2\pi} K(re^{i\theta})e^{u(re^{i\theta})}d\theta > \frac{C}{r\pi}\int_0^{2\pi} e^{u(re^{i\theta})}d\theta \ge \frac{C}{r}e^{I(r)}, \tag{3.3}$$

where we used the curvature bound assumption and Jensen's inequality. Differentiating yields

$$r^2I'' + rI' \ge Ce^I, \tag{3.4}$$

for r sufficiently large. Now note that

$$\Upsilon' = I'e^I, \quad \Upsilon'' = \left((I')^2 + I''\right)e^I. \tag{3.5}$$

Since $I' \to \infty$, for r large enough, we have $r^2(I')^2 \ge rI'$. Therefore $r^2I'' + r^2(I')^2 \ge Ce^{I(r)} \ge 0$ which shows $\Upsilon'' > 0$, i.e. Υ is a convex function. We substitute

$$I' = \frac{\Upsilon'}{\Upsilon}, \quad I'' = \frac{\Upsilon\Upsilon'' - (\Upsilon')^2}{\Upsilon^2} \le \frac{\Upsilon''}{\Upsilon}, \tag{3.6}$$

into differential inequality (3.5) to find

$$r^2\Upsilon'' + r\Upsilon' \ge C\Upsilon^2. \tag{3.7}$$

Now note that Υ and Υ' satisfy the hypotheses of Corollary 3. Therefore for every $\alpha > 0$ and $s > 0$ there is an $\tilde{r}$ with

$$\tilde{r}\Upsilon'(\tilde{r}) < s\Upsilon^{\alpha}(\tilde{r}), \quad \tilde{r}\Upsilon''(\tilde{r}) < s(\Upsilon'(\tilde{r}))^{\alpha}. \tag{3.8}$$

Combining these estimates, we find

$$\tilde{r}\Upsilon'' < s\left(\frac{s}{\tilde{r}}\Upsilon^{\alpha}\right)^{\alpha} = \frac{s^{\alpha+1}}{\tilde{r}^{\alpha}}\Upsilon^{\alpha^2} < s^{\alpha+1}\Upsilon^{\alpha^2}, \tag{3.9}$$

since $\tilde{r} > 1$. Now note that the inequality $\Upsilon < \Upsilon^2$ (which holds for large r) implies

$$\tilde{r}^2\Upsilon''(\tilde{r}) + \tilde{r}\Upsilon'(\tilde{r}) < s^{\alpha+1}\Upsilon^{\alpha^2}(\tilde{r}) + s\Upsilon^{\alpha}(\tilde{r}) < (s + s^{\alpha+1})\Upsilon^{\alpha^2}. \tag{3.10}$$

Fixing $\alpha = \sqrt{2}$ and $s + s^{\alpha+1} = C$ we find

$$\tilde{r}^2\Upsilon''(\tilde{r}) + \tilde{r}\Upsilon'(\tilde{r}) < C\Upsilon^2(\tilde{r}), \tag{3.11}$$

which contradicts inequality (3.7). Therefore $K > -C/2|z|^2$, which proves the theorem.

4. Examples and Discussion

We now give several examples which provide intuition for the theorems and demonstrate their limitations. If $g = e^u|dz|^2$ is a metric on $\mathbb{C}$ one can choose u to be different subharmonic functions so that g has negative Gaussian curvature. First, consider $u = a + b|z|^{2n}$ where $b, n > 0$. Then $\Delta u = 4bn^2|z|^{2(n-1)} > 0$ and u is of infinite order, i.e. $\beta = \infty$. We compute $K = -C|z|^{2(n-1)}e^{-b|z|^{2n}}$ where $C > 0$, which satisfies the asymptotic curvature bound of our second theorem. We can also generalize this example by taking a superposition over n, i.e.

$$\tilde{u} = a + b\int_0^{\Lambda} |z|^{2\alpha} d\alpha. \tag{4.1}$$

is also a subharmonic function of infinite order whose associated metric satisfies the asymptotic curvature bounds of Theorem 2.

Now consider $u = \ln([c + |z|^{2a}]^{1/b})$ for $0 < a, b, c < \infty$. Then

$$\Delta u = \frac{4a^2c|z|^{2a-2}}{b(c + |z|^{2a})^2} > 0, \tag{4.2}$$

and $\beta = 2a/b$. In particular,

$$K = -\frac{C|z|^{2a-2}}{(c + |z|^{2a})^{2-b^{-1}}}, \tag{4.3}$$

and thus asymptotically we have

$$K \approx -C|z|^{(2a-2)-(4a+2a/b)} = \frac{-C}{|z|^{\beta(b+1)+2}}. \tag{4.4}$$

Since $a > 0$ can be made arbitrarily small, this example shows Theorem 1 is false for $s > \beta + 2$. However, note that

$$\lim_{z\to\infty} |z|^{\beta+2}K(z) = 0\,,$$

i.e. the sequence statement in the theorem remains valid in this case. It would be interesting to find an example where u is of finite order β and

$$\liminf_{z\to\infty} |z|^{\beta+2}K(z) > 0\,.$$

The properties of solutions to the prescribed Gauss curvature equation on the plane where were studied in [2, 3, 4]. Specifically, suppose $K \leq 0$ in $\mathbb{R}^2$ and $K(z) \sim -|z|^{-l}$ near ∞ for some constant $l > 2$. Then for each $\alpha \in (0, \frac{l-2}{2})$, they show (1.1) possesses a unique solution

$$u_\alpha(z) = \alpha \ln |z| + O(1), \quad \text{near}\, \infty. \tag{4.5}$$

The function U given by

$$U(x) = \sup\{u(x)|\, u \text{ is a solution of (1.1) in} \mathbb{R}^2\}$$

is well-defined on $\mathbb{R}^2$ and is also a solution of (1.1) in $\mathbb{R}^2$. Moreover, if u is an arbitrary solution of (1.1) in $\mathbb{R}^2$, then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \frac{l-2}{2})$.

If u is an arbitrary solution of (1.1) in $\mathbb{R}^2$. Then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \frac{l-2}{2})$. If $\frac{l-2}{2} > \alpha > \beta$, then $u_\alpha > u_\beta$ in $\mathbb{R}^2$. Furthermore, the asymptotic behavior of the maximal solution U near ∞ is given by

$$U(z) = \frac{l-2}{2} \ln |z| - \ln(\ln |z|) + O(1)\,. \tag{4.6}$$

This solution has order $\beta = \frac{l-2}{2}$ while the previous has order α.

It is interesting to note the constraints Corollary 1 puts on metrics of constant negative curvature. Given a C^2 metric g on $\mathbb{R}^2$ there is $C^{2,\alpha}$ diffeomorphism $w = \phi(z)$ from $\mathbb{R}^2$ onto $\mathbb{R}^2$ or D such that g is conformally flat in the $w, \bar{w}$ coordinates c.f. [1]. If $g = e^{u(w,\bar{w})}|dw|^2$ on the plane and has of constant negative curvature, then Corollary 1 requires $u(z, \bar{z})$ cannot be defined on the entire complex plane. One interesting implication of this statement is that there is no model of hyperbolic geometry that can be defined on the entire plane.

Finally, we note that there is no analogue of Corollary 1 in higher dimensions. Consider a conformally flat metric $g = u^{4/(n-2)}\delta$, where δ is the Euclidean metric, with $u > 0$ on $\mathbb{R}^n$ for $n \geq 3$. Then the scalar curvature R_g of g is given by

$$R_g = -4\frac{n-1}{n-2}u^{\frac{n-2}{n+2}}\Delta_g u. \tag{4.7}$$

Thus if we take $u(x) = |x|^2$, then we see that $R_g(x) \to -\infty$ in any direction as $|x| \to \infty$, so Corollary 1 does not generalize.

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