

LOCALLY ISOMETRIC FAMILIES OF MINIMAL SURFACES

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ABSTRACT. We consider a surface M immersed in \mathbb{R}^3 with induced metric $g = \psi\delta_2$ where δ_2 is the two dimensional Euclidean metric. We then construct a system of partial differential equations that constrain M to lift to a minimal surface via the Weierstrauss-Enneper representation demanding the metric is of the above form. It is concluded that the associated surfaces connecting the prescribed minimal surface and its conjugate surface satisfy the system. Moreover, we find a non-trivial symmetry of the PDE which generates a one parameter family of surfaces isometric to a specified minimal surface. We demonstrate an instance of the analysis for the helicoid and catenoid.

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1. BACKGROUND

1.1. Harmonic Mapping Preliminaries. Given two C^2 functions u and v that satisfy Laplace's equation, a complex valued harmonic function f is defined via the combination: $f = u + iv$. The Jacobian of such a function is given by $J_f = u_x v_y - u_y v_x$. We will only consider harmonic functions that are univalent (injective) with positive Jacobian on $\mathbb{D} = \{z : |z| < 1\}$. On a simply connected domain $D \subset \mathbb{C}$, a harmonic mapping f has a canonical decomposition $f = h + \bar{g}$ where h and g are analytic in D , which is unique up to a constant. The dilatation ω of a harmonic map f is defined by $\omega \equiv g'/h'$. The following theorem provides the link between harmonic univalent functions and minimal surfaces:

Theorem 1. (*Weierstrass-Enneper Representation*). *Every regular minimal surface has locally an isothermal parametric representation of the form*

$$(1.1) \quad \left(\operatorname{Re} \left\{ \int^z p(1+q^2)dw \right\}, \operatorname{Im} \left\{ \int^z p(1-q^2)dw \right\}, 2\operatorname{Im} \left\{ \int^z pqdw \right\} \right)$$

in some domain $D \subset \mathbb{C}$, where p is analytic and q is meromorphic in D , with p vanishing only at the poles (if any) of q and having a zero of precise order $2m$ wherever q has a pole of order m . Conversely, each such pair of functions p and q analytic and meromorphic, respectively, in a simply connected domain D generate through the formulas (1.1) an isothermal parametric representation of a regular minimal surface.

We will use (1.1) in the following form:

Corollary 1. *For a harmonic function $f = h + \bar{g}$, define the analytic functions h and g by $h = \int^z pd\zeta$ and $g = -\int^z pq^2d\zeta$. Then the minimal surface representation (1.1) becomes*

$$(1.2) \quad \left(\operatorname{Re}\{h+g\}, \operatorname{Im}\{h-g\}, 2\operatorname{Im} \left\{ \int^z \sqrt{h'g'}d\zeta \right\} \right)$$

See [6], [3] for a further introduction to harmonic mappings. For an introduction to minimal surfaces see [5].

2. THE ISOMETRIC CONDITION

Let $\mathbf{x}(u, v)$ be a parametrization for a surface M immersed in \mathbb{R}^3 . Set $z = u + iv$ and define $\phi = \partial\mathbf{x}/\partial z$. Let $E, F,$ and G be the coefficients of the metric induced in \mathbb{R}^3 by $\mathbf{x}(u, v)$. We then have the relations

$$(2.1) \quad \phi^2 = \frac{1}{4}(E - G - 2iF)$$

$$(2.2) \quad \bar{\phi}^2 = \frac{1}{4}(E - G + 2iF)$$

$$(2.3) \quad |\phi|^2 = \frac{1}{4}(E + G)$$

where ϕ^2 is notation for $\phi \cdot \phi$. Inverting this system we find

$$\begin{aligned} E &= \bar{\phi}^2 + \phi^2 + 2|\phi|^2 \\ F &= i(\phi^2 - \bar{\phi}^2) \\ G &= -\bar{\phi}^2 - \phi^2 + 2|\phi|^2 \end{aligned}$$

Since the Weierstrauss-Enneper representation theorem requires that M has an isothermal parametrization, we require $E = G$ and $F = 0$, which implies

$$\phi^2 = 0 \quad \bar{\phi}^2 = 0 \quad E = 2|\phi|^2$$

The first two equations are identically satisfied. Expanding the constraint on E , and using the identity

$$|\phi|^2 = \frac{1}{4}|p|^2 \left((1 + q^2)(1 + \bar{q}^2) + (1 - q^2)(1 - \bar{q}^2) + 4q\bar{q} \right)$$

we find $E = |h'|^2 + |g'|^2$. Defining $\text{Re}\{h\} = {}_1h$, $\text{Im}\{h\} = {}_2h$, $\text{Re}\{g\} = {}_1g$, and $\text{Im}\{g\} = {}_2g$, we have the Cauchy Riemann and isometric conditions:

$$(2.4) \quad {}_1h_u - {}_2h_v = 0 \quad {}_1h_v + {}_2h_u = 0$$

$$(2.5) \quad {}_1g_u - {}_2g_v = 0 \quad {}_1g_v + {}_2g_u = 0$$

$$(2.6) \quad {}_1h_u^2 + {}_2h_u^2 + {}_1g_u^2 + {}_2g_u^2 - 2\sqrt{({}_1h_u^2 + {}_2h_u^2)({}_1g_u^2 + {}_2g_u^2)} = 0$$

We now proceed to calculate the symmetry group of (2.4)-(2.6). For an introduction to symmetry methods see [2], [7], and [8].

3. SYMMETRY ANALYSIS

We now turn attention to (2.4)-(2.6). The infinitesimal generator of the above system is given by:

$$\mathbf{v} = c^u \partial_u + c^v \partial_v + c^{1h} \partial_{1h} + c^{2h} \partial_{2h} + c^{1g} \partial_{1g} + c^{2g} \partial_{2g} + c^\psi \partial_\psi.$$

Since the system is first order, we need only consider the first prolongation

$$\begin{aligned} \text{pr}^{(1)}\mathbf{v} &= \mathbf{v} + {}_1h^u \partial_{1h_u} + {}_1h^v \partial_{1h_v} + {}_2h^u \partial_{2h_u} + {}_2h^v \partial_{2h_v} \\ &\quad + {}_1g^u \partial_{1g_u} + {}_1g^v \partial_{1g_v} + {}_2g^u \partial_{2g_u} + {}_2g^v \partial_{2g_v} \end{aligned}$$

where the c^i are functions of $u, v, \psi, {}_1h, {}_2h, {}_1g$, and ${}_2g$. Applying the first prolongation to the PDE system gives the following coefficient relationships:

$$\begin{aligned} {}_1h^u - {}_2h^v &= 0 & {}_1h^v + {}_2h^u &= 0 \\ {}_1g^u - {}_2g^v &= 0 & {}_1g^v + {}_2g^u &= 0 \\ {}_1h_u {}_2h^u + {}_2h_u {}_2h^u + {}_1g_u {}_1g^u + {}_2g_u {}_2g^u - c^\psi &= 0 \end{aligned}$$

For the first part of the symmetry analysis, we will only consider the Cauchy Riemann coefficient conditions. The prolongation formula gives

$$\begin{aligned} i h^\alpha &= D_\alpha c^{ih} - i h_u D_\alpha c^u - i h_v D_\alpha c^v \\ i g^\alpha &= D_\alpha c^{ig} - i g_u D_\alpha c^u - i g_v D_\alpha c^v \end{aligned}$$

where $i \in \{1, 2\}$ and $\alpha \in \{u, v\}$. Substituting into the coefficient formulas and equating coefficients gives the following simplified system of PDE

$$c_u^{1h} = c_v^{2h} \quad c_{1h}^{1h} = c_u^u \quad c_{2h}^{1h} = -c_v^u \quad c_{1h}^{2h} = -c_u^v \quad c_{2h}^{2h} = c_v^v$$

$$\begin{aligned}
 c_u^{1g} &= c_v^{2g} & c_{1g}^{1g} &= c_u^u & c_{2g}^{1g} &= -c_v^u & c_{1g}^{2g} &= -c_u^v & c_{2g}^{2g} &= c_v^v \\
 c_v^{1h} &= -c_u^{2h} & c_{1h}^{1h} &= c_v^v & c_v^u &= c_{1h}^{2h} & c_{2h}^{1h} &= c_u^v & c_{2h}^{2h} &= c_u^u \\
 c_v^{1g} &= -c_u^{2g} & c_{1g}^{1g} &= c_v^v & c_v^u &= c_{1g}^{2g} & c_{2g}^{1g} &= c_u^v & c_{2g}^{2g} &= c_u^u
 \end{aligned}$$

as well as the six conditions:

$$\begin{aligned}
 c^u &= c^u(u, v) & c^v &= c^v(u, v) \\
 c^{1h} &= c^{1h}(u, v, {}_1h, {}_2h) & c^{2h} &= c^{2h}(u, v, {}_1h, {}_2h) \\
 c^{1g} &= c^{1g}(u, v, {}_1g, {}_2g) & c^{2g} &= c^{2g}(u, v, {}_1g, {}_2g)
 \end{aligned}$$

We note that these equations represent Cauchy Riemann conditions on the pair c^u and c^v as well as the generalized analytic functions given by $c^h = c^{1h} + ic^{2h}$ and $c^g = c^{1g} + ic^{2g}$. We now turn attention to the isometric condition. Performing the same analysis yields six redundant equations and the new condition

$$0 = c_u^v = c^{\psi} = c_u^{1h} = c_u^{2h} = c_u^{1g} = c_u^{2g}$$

Using (2.4), (2.5) to change the u derivatives to v derivatives in (2.6) and then repeating the above argument gives analogous conditions with $u \rightarrow v$. Thus we find the infinitesimal generator coefficients are of the form:

$$\begin{aligned}
 c^u &= c^u(u) & c^v &= c^v(v) & c^{\psi} &= 0, & c^{1h} &= c^{1h}({}_1h, {}_2h) \\
 c^{2h} &= c^{2h}({}_1h, {}_2h) & c^{1g} &= c^{1g}({}_1g, {}_2g) & c^{2g} &= c^{2g}({}_1g, {}_2g)
 \end{aligned}$$

The condition $c_u^u = c_v^v$ implies $c^u = c_1u + c_2$ and $c^v = c_1v + c_3$. Other similar conditions show that all the coefficients are affine with c^1 as the linear coefficient. Setting all but one constant to zero gives the following symmetries:

$$\begin{aligned}
 \mathbf{v}_1 &= \partial_u & \mathbf{v}_2 &= \partial_v & \mathbf{v}_3 &= \partial_{1h} & \mathbf{v}_4 &= \partial_{2h} & \mathbf{v}_5 &= \partial_{1g} & \mathbf{v}_6 &= \partial_{2g} \\
 \mathbf{v}_7 &= u\partial_u + v\partial_v + {}_1h\partial_{1h} + {}_2h\partial_{2h} + {}_1g\partial_{1g} + {}_2g\partial_{2g}
 \end{aligned}$$

Exponentiating these infinitesimal vector fields gives the solutions:

$$(3.1) \quad h \rightarrow h(z - s) \quad g \rightarrow g(z - s)$$

$$(3.2) \quad h \rightarrow h(z - is) \quad g \rightarrow g(z - is)$$

$$(3.3) \quad h \rightarrow h + s \quad g \rightarrow g$$

$$(3.4) \quad h \rightarrow h + is \quad g \rightarrow g$$

$$(3.5) \quad h \rightarrow h \quad g \rightarrow g + s$$

$$(3.6) \quad h \rightarrow h \quad g \rightarrow g + is$$

$$(3.7) \quad h \rightarrow e^s h(e^{-s}z) \quad g \rightarrow e^s g(e^{-s}z)$$

In [1] the minimal symmetry group for the real minimal surface equation

$$u_{xx}(1 + u_y^2) + u_{yy}(1 + u_x^2) - 2u_x u_y u_{xy} = 0$$

was calculated. Many of the translational symmetries and an $e^s f(e^{-s}x, e^{-s}y)$ symmetry were found. We note that the complex analogue in \mathbf{v}^7 is similar but different, since it constrains a Weierstrass-Enneper representation of a surface and not a graph.

4. SYMMETRY COMMENTS

Consider the transformation $h \rightarrow e^{i\theta}h$, $g \rightarrow e^{i\theta}g$ which preserves the metric $E = |h'|^2 + |g'|^2$. When $\theta = 0$, this is simply a minimal surface specified by defining ψ . When $\theta = \pi/2$ we get the conjugate surface. Thus all intermediate surfaces, called associated surfaces, are isometric. We do however note that this transformation is not a symmetry of the system since there is no transformation $h \rightarrow e^{i\theta}h$, leaving h real valued. Since all minimal surfaces can be constructed from parts of a helicoid and catenoid [4], the following examples are of interest. First we draw attention to the catenoid, given by $\psi = \cosh(v)^2$. Its conjugate surface is the helicoid and associated surfaces between the two are plotted over \mathbb{D} in Figure 1. Since all of the associated surfaces are isometric, geometrically they are equivalent. However, note topologically the catenoid is $S^1 \times \mathbb{R}$ where as the helicoid is \mathbb{R}^2 .

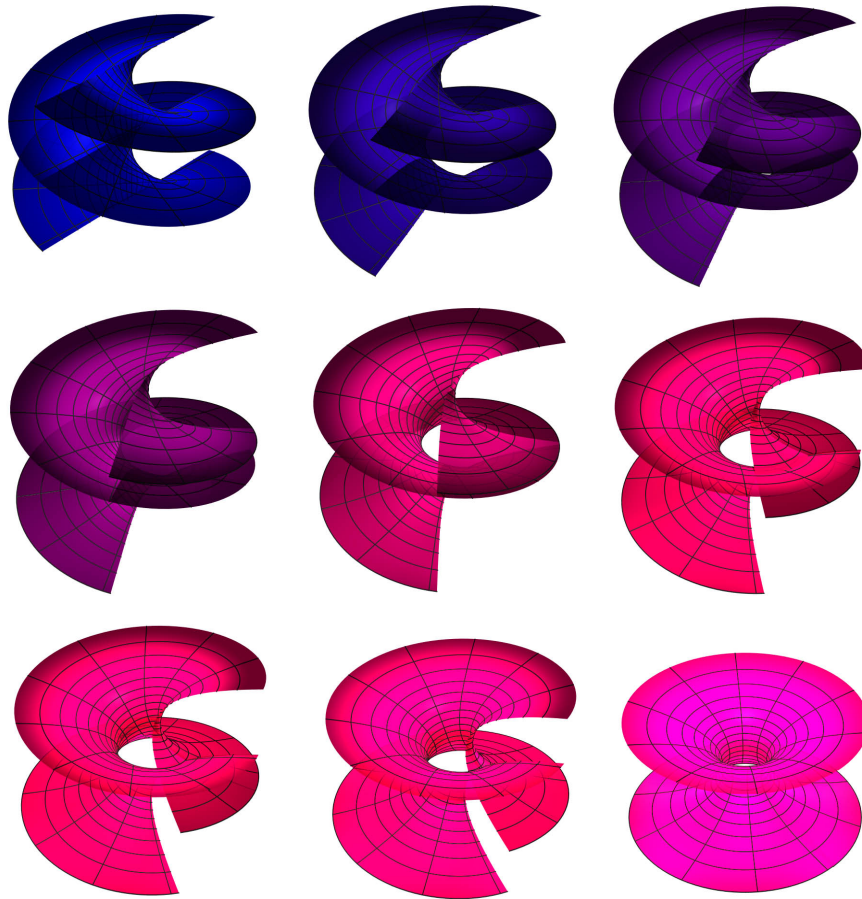


FIGURE 1. Helicoid to Catenoid Transformation.

We now turn our attention to the other symmetries found in the analysis for the half catenoid. We will see that the symmetries generate surfaces that are topologically distinct from the catenoid, but geometrically identical as in the above example. Let f be the harmonic mapping $f = h + \bar{g}$ where

$$h = \frac{1}{2} \left(\frac{1}{2} \log \left[\frac{1+z}{1-z} \right] + \frac{z}{1-z^2} \right)$$

and

$$g = \frac{1}{2} \left(\frac{1}{2} \log \left[\frac{1+z}{1-z} \right] - \frac{z}{1-z^2} \right)$$

which lifts to the catenoid. We make the transformation in equation (3.7) by letting $h \rightarrow e^s h(e^{-s}z)$ and $g \rightarrow e^s g(e^{-s}z)$. Figure 2 gives several plots of this transformation for various s values. The topology of the half catenoid is \mathbb{R}^2 for all s up to some value between (0.3,0.4) where it changes to a punctured cylinder. Note as $s \rightarrow \infty$ that the minimal surfaces eventually degenerate to a line, in a manner peculiarly similar to neckpinch singularities of the Ricci Flow. We note symmetry (3.2) is a scaled rotation, and can not comment on (3.1) in this example. The rest of the symmetries are translations.

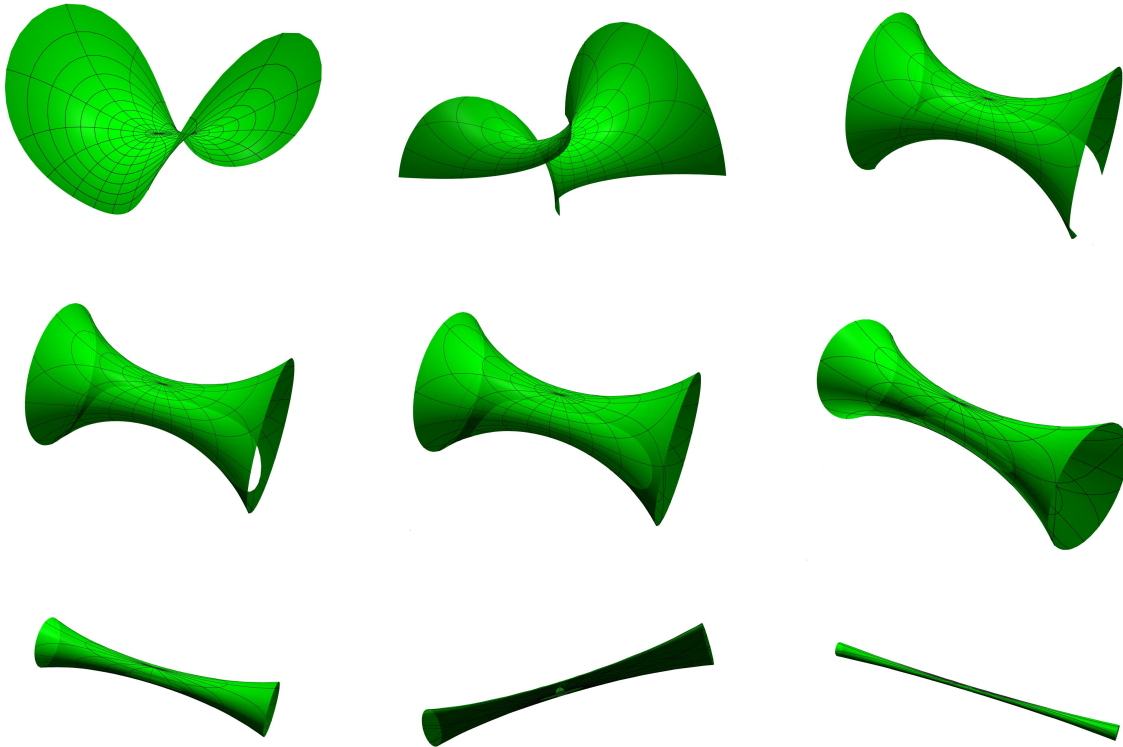


FIGURE 2. Symmetry (3.7) for $s = \{-1.2, -0.5, 0, 0.3, 0.4, 0.5, 1, 1.5, 3\}$.

When (3.7) is applied to the helicoid, we find that the number of rotations of the helicoid about its axis are scaled. Thus we have:

Theorem 2. *Let S be the helicoid over \mathbb{D} parameterized isothermally by $\mathbf{x} = (\sinh u \sin v, \sinh u \cos v, -v)$. For helicoids S_1 given by $u \in (0, 2\pi)$, $v \in (v_0, v_1)$, and S_2 by $u \in (0, 2\pi)$, $v \in (v_2, v_3)$ where $v_i \in \mathbb{R}$ then S_1 and S_2 are locally isometric.*

It would be interesting to generalize the symmetry methods of this paper to higher dimensional Riemannian or Lorentzian manifolds. One would need a generalized Weierstrauss-Enneper which we are not aware exists. Moreover, we believe there are potential topological theorems coming from symmetry (3.7). For instance, if one calculates the one parameter family of minimal surfaces given by symmetry (3.7) and a simply connected minimal surface, does the topology always change from the plane to $S^1 \times \mathbb{R}^1$ or some variant thereof?

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