A NOVEL REDUCTION OF THE SIMPLE ASIAN OPTION AND LIE-GROUP INVARIANT SOLUTIONS

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Abstract. We develop the complete 6-dimensional classical symmetry group of the partial differential equation (PDE) that governs the fair price of a simple Asian option within a simple market model. The symmetries we expose include the 5-dimensional symmetry group partially noted by Rogers and Shi, and communicated implicitly by the change of numéraire arguments of Večeř (in which symmetries reduce the original 2+1 dimensional simple Asian option PDE to a 1+1 dimensional PDE). Going beyond this previous work, we expose a new 1-dimensional space of symmetries of the Asian PDE that cannot reasonably be found by inspection. We demonstrate that the new symmetry could be used to formulate a new, "nonlinear" derivative security that has a 1+1 dimensional PDE formulation. We indicate that this nonlinear security has a closed-form pricing formula similar to that of the Black-Scholes equation for a particular market dependent payoff, and show that hedging the short position in this particular exotic option is stable for all market parameters. We also demonstrate the patently Lie-algebraic method for obtaining the already well-known "Rogers-Shi-Večeř" reduction.

Simple Asian Option, Symmetry Analysis, Rogers-Shi-Večeř reduction.

1. Introduction

An Asian option is an option whose payoff depends on a time average of the price of an underlying asset. Here we discuss one of the simplest versions of this option within one of the simplest market models, namely that with a payoff based on the time-continuous, arithmetic average value, at a predetermined expiration date, of a risky asset undergoing geometric Brownian motion. Currently there is no known formula for the pricing of such an Asian option before expiration for all values of its underlying asset. Since the payoff is path dependent in the state space of only the underlying asset price, but not in that state space augmented with the running average of the asset, the PDE relevant to this pricing problem naturally comes in two space dimensions [4]. Unfortunately the absence of diffusion and the presence of hyperbolicity in the direction of the average asset value in this state space make it so that, in certain market regimes, oscillations are prone to occur in those approximate solutions of this PDE which are generated by simple minded numerical schemes. Consequently there has been a significant amount of work devoted to developing sophisticated numerical methods for this pricing problem [8], [9],[14]. On the other hand, in [1] and [2], Večeř used change of numéraire and other financially insightful arguments to reduce the pricing problem of the Asian option to valuing a certain simple, *non time-averaged* claim: in either of the works indicated, the claim is on the asset-denominated, final value of, respectively, certain non self-financing and self-financing portfolios of risky and risk-free assets (which we will call stock and bond). Večeř's approach included the evidently powerful idea of demanding that these deterministic, stock-denominated portfolios replicate the Asian forward. In his works the relevant parabolic, 1 space variable PDE's obtained are robustly simulated with simple numerical schemes, particularly in the latter treatise where the equation is purely diffusive and, so, even the simplest of schemes are unconditionally stable. Rogers and Shi found a similar reduction to a 1-dimensional state space in [10] previous to the work of Večeř, where they used a certain evident invariance of the Asian option equation under a homogeneous change of scale in the original 2 dimensional state space. (The resulting PDE may be difficult to numerically simulate, however, depending on market parameters, without advecting away certain deterministic processes that amount to implementing one or the other of Večeř's reductions.) We demonstrate how one may arrive at any of these various reductions, including all other possible "point" reductions, in a systematic fashion, i.e. by the methods of Sophus Lie's symmetry analysis [11]. In addition to finding the symmetry used explicitly by Rogers and Shi, and demonstrated implicitly by Večeř, we find additional symmetries of the Asian option equation. One such symmetry leads to a new, nontrivial reduction of the equation.

2. The Classical Asian Option Reduction

We first review the usual simple market model for the type of Asian option to be analyzed. Let $S(t)$ denote the price process of a stock undergoing geometric Brownian motion, i.e. let $S(t)$ satisfy the stochastic differential equation (SDE)

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t). \tag{1}$$

Here $\tilde{W}(t)$, $0 \le t \le T$ is a 1-dimensional standard Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$, so that r denotes the deterministic rate of return of a risk-free bond in this simple market. Let $K \ge 0$ be the strike price of a fixed-strike Asian call option whose payoff at time T is given by

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+. \tag{2}$$

Within the indicated model, the no-arbitrage and perfectly hedgable price $V(t)$ at which such an Asian call can be safely traded (with no injury to buyer or seller) at time t since initiation of the contract is given by

$$V(t) = v(x(t), y(t), T - t), \tag{3}$$

where $x(t) = S(t)$ is the value of the asset underlying the derived security at time t, $y(t)/t$ is the arithmetic mean of the asset value over $[0, t] \subset [0, T]$, i.e.

$$\frac{y(t)}{t} = \frac{1}{t}\int_0^t x(u)du = \frac{1}{t}\int_0^t S(u)du, \tag{4}$$

and where $v = v(x, y, \tau)$ is the solution of the following initial value problem:

$$\left(-\partial_\tau - r + rx\partial_x + x\partial_y + \frac{1}{2}\sigma^2 x^2 \partial_x^2\right) v(x, y, \tau) = 0, \tag{5}$$

$$v(x, y, 0) = \max\left\{\frac{y}{T} - K, 0\right\}. \tag{6}$$

As per (3), in (5) we have $\tau = \tau(t) = T - t$, i.e. $\tau(t)$ is the time remaining until expiration of the contract when t is the time since its initiation. We will refer to equation (5) as the Asian equation. The numéraire associated with the Asian equation is that of the strike price K of the indicated option (dollars say), and, as per (6) and (4), the value process is evidently the running average of the stock. The initial value problem (5), (6), should include not only the initial data (6) (or said terminal data in terms of t) but also the domain of validity of the PDE ($x > 0$, $y \in \mathbb{R}$, $\tau \in (0, T]$), as well as information about the values of $v(x, y, \tau)$ as $x \to 0$ and $y \to -\infty$. See [12, p. 322-323]. The latter boundary conditions encode information that may be gotten directly from the representation of the price as a risk neutral expectation of the payoff (2), but need not concern us here because they are natural to the dynamical system generated by the structure of the PDE, a large class of initial data (including the one in (6)), the indicated domain of validity of the PDE and a large but relevant function space in which the initial data lie; the unstated boundary conditions can be deduced from the equation, benign initial data prescribed within the indicated domain of validity and exclusion of pathological growth for the solution itself. This will manifest itself very practically here, namely by the fact that the relevant boundary conditions preserve the symmetries of the payoff. See ([6, p.209,211,217]) for similar uniqueness results.

For a modern review of Asian options see for example [10],[12, p. 320-330]. For a review giving insight into why Asian and other options not written solely on market tradables are, in a certain sense, difficult to price, see [3, ch. 1,3].

In [2], Večeř reduced the Asian initial value problem (5), (6) to the two variable initial value problem

$$\left(-\partial_\tau + \frac{1}{2}\sigma^2 (z - \Delta_X(\tau))^2 \partial_z^2\right) g(z, \tau) = 0, \quad z \in \mathbb{R}, \quad \tau \in [0, T], \tag{7}$$

$$g(z, 0) = z^+, \quad z \in \mathbb{R}, \tag{8}$$

where

$$\Delta_X(\tau) = \frac{1 - e^{-r\tau}}{rT}. \tag{9}$$

As indicated in the introduction, the value process of the reduced equation (7) is that of a self-financing portfolio of stock and bond replicating the associated Asian forward, this value denominated in the numéraire of the stock. That this is so can be understood to some extent by noting that, as shown in [2], the associated fair price of the Asian call is then given in terms of $g = g(z, \tau)$ satisfying (7), (8) through

$$V(t) = v(x(t), y(t), T-t) = x(t) g\left(\frac{X(x(t), y(t), T-t)}{x(t)}, T-t\right), \tag{10}$$

where $x(t)$ and $y(t)$ are related to the stock evolution as in (3), and where

$$X(x, y, \tau) = \Delta_X(\tau)x + e^{-r\tau}(y/T - K) : \tag{11}$$

In [2] it is shown that $X(x, y, T-t)$ is the value of a portfolio holding $\Delta_X(T-t)$ shares of the stock at any given time $t \in [0, T]$, capital in the amount of $\Delta_X(T-0)x(0) + e^{-r(T-0)}(y(0)/T - K) = \Delta_X(T)S(0) - e^{-rT}K$ having been borrowed at time $t = 0$ from the money market to finance the initial stock investment. And since

$$\begin{aligned} X(x(T), y(T), T-T) &= \Delta_X(T-T)x(T) + e^{-r(T-T)}\left(\frac{1}{T}y(T) - K\right) \\ &= \frac{1}{T}\int_0^T S(u)du - K, \end{aligned} \tag{12}$$

the value of the associated forward contract has been replicated by time T. Importantly the choice (11) for the replicating portfolio (essentially $\Delta_X(\tau)$ and the initial capital) is entirely independent of the market dynamics (1).

As shown below, the reduction that Rogers and Shi produced is related to (7) in a simple way. Consequently we will call either reduction the Rogers-Shi-Večeř reduction. We will now show that this reduction arises naturally as a Lie group invariant solution of the Asian equation. Moreover, we will find an additional class of group invariant solutions which leads to a novel reduction of equation (5).

3. Symmetry Analysis

A point symmetry of a differential equation is a change of variables, both dependent and independent, under which the equation is invariant. For instance, the change of variables $(x, t, u) \to (\lambda x, \lambda^2 t, u)$ is a point symmetry of the one-dimensional heat equation $\partial_t u(x, t) = \partial_{xx} u(x, t)$. The collection of all point symmetries of an equation forms a local differentially varying group called the symmetry group of the equation. Such a group is also called a local Lie group. The symmetry group may be identified with a manifold (generalized surface) because of its Lie group structure [11]. In particular, the Lie group and hence the manifold can be thought of as generated locally from the union of integral curves of a system of vector fields defined on the space of dependent and independent variables. The elements of any such set of vector fields generating the group in this local fashion are called infinitesimal generators of the group. The infinitesimal generators of the group of symmetries form a vector space which together with a bilinear commutator operation define an algebra—specifically a Lie algebra—on the tangent space of the manifold. Thus the Lie algebra gives local geometric information about the structure of the manifold.

The proper setting for discussing the theory of symmetry groups of differential equations is within Lie group theory and manifold theory; however, it is possible to avoid referring to these subjects (at a cost of some geometric intuition and motivation) and still calculate the symmetries of an equation. There are several variations of a pseudo-algorithmic method that allows one to compute the symmetry group of a differential equation. We summarize the mathematically rigorous version in the Appendix and refer the interested reader to [11] for more information. We will construct the symmetry group of the Asian equation with a simpler but equivalent method. This will essentially involve coupling a first order equation, called the symmetry equation, to the Asian equation with an additional variable quantifying motion in the space of symmetries, and then solving the first order symmetry equation. We will use the solution to generate exact solutions of the Asian equation. We first demonstrate this method with the simpler heat equation.

Suppose $u = u(x, t)$ solves the heat equation

$$\partial_t u = \partial_x^2 u, \tag{13}$$

and demand its extension $u = u(s, x, t)$ solves both the heat equation (13) and the symmetry equation

$$\partial_s u = -(T(t, x, u)\partial_t + X(t, x, u)\partial_x)u + U(t, x, u). \tag{14}$$

Note here that T, X, and U are indicated to be functions of only the independent variables t and x, and the dependent variable u, of the heat equation (13). They are not assumed to be functions of any of the derivatives of the latter with respect to the former, such as u_{xxx}, u_{xx}, u_x, or u_t $(= u_{xx})$. Thus it turns out that commutativity of partial derivatives implies the compatibility condition

$$\begin{aligned} 0 &= [\partial_s, \partial_t]u \equiv (\partial_s \partial_t - \partial_t \partial_s)\, u \\ &= -2u_{xxx}(u_x T_u + T_x) + u_{xx}\big(T_t - u_x^2 T_{uu} - 2u_x\,(X_u + T_{xu}) - 2X_x - T_{xx}\big) \\ &\quad + u_x\left(-u_x^2 X_{uu} + u_x(U_{uu} - 2X_{xu}) + 2U_{xu} + X_t - X_{xx}\right) + U_{xx} - U_t. \end{aligned} \tag{15}$$

Now since T, X, and U are independent of u_{xxx}, u_{xx}, u_x, equation (15) is satisfied over the space of smooth solutions of the heat equation (13) if and only if

$$0 = T_u = T_x = X_u = U_{uu} = T_t - 2X_x = 2U_{xu} + X_t = U_{xx} - U_t. \tag{16}$$

Integrating the simple linear system of PDE's (16) we find the general solution

$$\begin{aligned} \begin{pmatrix} T \\ X \\ U \end{pmatrix} &= \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2t \\ x \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4t^2 \\ 4tx \\ -(2t + x^2)u \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 2t \\ -xu \end{pmatrix} \\ &+ \eta \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_0 \end{pmatrix}, \end{aligned} \tag{17}$$

where Greek letters denote arbitrary constants and $u_0 = u_0(x, t)$ is any fixed solution to the heat equation. If we set $\gamma = 1$ and all other parameters and u_0 to zero ($u_0 = 0$ is a solution to the heat equation), we find from (17) that the symmetry equation (14) becomes

$$\partial_s u = -\left(4tx\partial_x + 4t^2\partial_t + 2t + x^2\right) u, \tag{18}$$

whose general solution $u = u(s, t, x)$ can be expressed as

$$u(s, t, x) = \frac{1}{(1 + 4st)^{1/2}} \exp\left(-\frac{sx^2}{1 + 4st}\right) u\left(0, \frac{t}{1 + 4st}, \frac{x}{1 + 4st}\right). \tag{19}$$

Thus in (19) one has a family of smooth solutions $u(s, t, x)$ of the heat equation parameterized by the symmetry coordinate s, each family member originating at $s = 0$ from any specific smooth solution $u(0, x, t)$ of the heat equation. If we take $u(0, t, x) = 1$, which is a simple solution to the heat equation (13), and apply the time translational invariance symmetry of the heat equation (associated with the parameter α in (17)), along with the linearity of the equation to permit rescaling a solution (which is associated with the parameter η in (17)), we find the fundamental solution

$$u(s, t, x) = \frac{1}{(4\pi t)^{1/2}} \exp(-x^2/4t), \tag{20}$$

for some choice of s (and a corresponding choice of the triple α, γ, and η). We now apply the same method to the Asian equation.

4. The Point Symmetries of the Asian Equation

Consider the following coupling of the Asian equation (5) to its associated symmetry equation:

$$\partial_\tau v = \left(-r + rx\partial_x + x\partial_y + \frac{1}{2}\sigma^2 x^2 \partial_x^2\right) v, \tag{21}$$

$$\begin{aligned} \partial_s v &= -\left(\mathcal{T}(\tau, x, y, v)\partial_\tau + X(\tau, x, y, v)\partial_x + Y(\tau, x, y, v)\partial_y\right) v \\ &+ \mathcal{V}(\tau, x, y, v). \end{aligned} \tag{22}$$

Requiring the compatibility condition $0 = [\partial_s, \partial_\tau]v$ we find simple linear PDE's describing $\mathcal{T}$,X,Y,and $\mathcal{V}$'s dependence on τ,x,y and v. These PDE's are similar in form to those stemming from the heat equation, namely those encountered in (16). The general solution to the simple linear PDE's arising from the compatibility condition $0 = [\partial_s, \partial_\tau]v$ can be expressed as

$$\begin{pmatrix} \mathcal{T} \\ X \\ Y \\ \mathcal{V} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 2\sigma^2 xy \\ \sigma^2 y^2 \\ 2(x - ry)v \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_0 \end{pmatrix}, \tag{23}$$

where $v_0 = v_0(\tau, x, y)$ is any fixed solution of the Asian equation (21), and α, β, γ, δ, ϵ are arbitrary constants.

We will see that the symmetry associated with the vector field parameterized by γ in (23) leads to a new option with exceptionally simple dynamics, and which is infinitely differentiable (C^∞) in all values of the relevant state variables, i.e. in all values of accumulation y and stock price x. In particular it will be C^∞ in these state variables even near and at the relevant notion of strike price and even near and at expiration. First though we demonstrate how Večeř's reduction (7) arises from the point symmetry group of the Asian equation (21), the infinitesimal generators of this group indicated exhaustively in (23). The reduction is realized by choosing

$$\alpha = \beta - 1 = \gamma = \delta + KT = \epsilon - 1 = u_0 = 0 \tag{24}$$

in (23). Importantly, in (24) we have chosen $\gamma = 0$ for use in (23), and thereby have avoided using the new symmetry. We relax this restriction in section 5 below.

In order to get Večeř's reduction (7), in addition to choosing the parameters in (23) as per (24), we seek an associated *group invariant solution* $v = v(s, \tau, x, y)$ of the Asian equation (21), i.e. one in which we demand that the trivial dynamics

$$\begin{aligned} \partial_s v(s,\tau,x,y) = \quad 0 \quad &= -\left(\mathcal{T}(\tau,x,y,v)\partial_\tau + X(\tau,x,y,v)\partial_x + Y(\tau,x,y,v)\partial_y\right) v \\ &+ \quad \mathcal{V}(\tau,x,y,v) \end{aligned} \tag{25}$$

associated with (22) hold in the space of symmetries, at least for all values of the symmetry coordinate s in a neighborhood of $s = 0$. Thus, instead of demonstrating a family of solutions parameterized by s that stem from any particular solution $v = v(0, \tau, x, y)$ of the Asian equation (21) (as in (19) for the heat equation (13)), rather we find special solutions $v = v(0, \tau, x, y)$ of the symmetry equation (22) for which $v(s, \tau, x, y) = v(0, \tau, x, y)$ holds uniformly in such a neighborhood of $s = 0$. The theory described indicates that there is a class of such "static" solutions of the symmetry equation (22), i.e. solutions of the "group invariance equation" (25), which will also be a class of (group invariant) solutions of the Asian equation. With the parameters in (23) chosen as per (24), one finds that the general solution $v(s, \tau, x, y) = v(0, \tau, x, y) = v(\tau, x, y)$ of the group invariance equation (25) is of the form

$$v(x, y, \tau) = xh(x^{-1}(y/T - K), \tau) \equiv xh(z, \tau), \tag{26}$$

where h is any sufficiently smooth function of its arguments. One confirms that there are solutions $v(x, y, \tau)$ of the Asian equation (21) of the form (26) by noting that

$$\begin{aligned} 0 &= \left(\partial_\tau - \left(-r + rx\partial_x + x\partial_y + \frac{1}{2}\sigma^2 x^2 \partial_x^2\right)\right) v \\ &= \left(\partial_\tau - \left(-r + rx\partial_x + x\partial_y + \frac{1}{2}\sigma^2 x^2 \partial_x^2\right)\right) xh(x^{-1}(y/T - K), \tau) \\ &= x\left(\partial_\tau - \left(L + \frac{1}{2}\sigma^2 z^2 \partial_z^2\right)\right) h(z, \tau), \end{aligned} \tag{27}$$

where $L \equiv (T^{-1} - rz)\partial_z$, and where the last step in (27) requires significant but straightforward manipulations. Thus we have found that a) there exists solutions of the Asian equation (21) of the form indicated in (26), and b) have found that such solutions satisfy the reduced equation

$$0 = \left(\partial_\tau - \left(L + \frac{1}{2}\sigma^2 z^2 \partial_z^2\right)\right) h(z, \tau), \tag{28}$$

which is closely related to the Večeř's reduction (7). We recover Večeř's reduction (7) precisely by composing away the evolution associated with the deterministic advection term L in (28), i.e. by defining $g = g(z, \tau)$ via

$$\begin{aligned} h(z,\tau) = \exp(\tau L \partial_z) g(z,\tau) &= g\left(e^{-r\tau} z + \frac{1 - e^{-r\tau}}{rT}, \tau\right) \qquad (29) \\ &= g\left(\frac{e^{-r\tau} zx + \Delta_X(\tau)x}{x}, \tau\right) \\ &= g\left(\frac{e^{-r\tau}(y/T - K) + \Delta_X(\tau)x}{x}, \tau\right) \\ &= g\left(\frac{X(x,y,\tau)}{x}, \tau\right). \qquad (30) \end{aligned}$$

Here we used Večeř's portfolio X replicating the Asian forward, as in (11), (9). Thus, with the relevant solution to the Asian equation (21) given by (26),(30), we get

$$v(x,y,\tau) = xg\left(\frac{X(x,y,\tau)}{x},\tau\right), \tag{31}$$

which is (10). Moreover, conjugating the deterministic evolution $\exp(\tau L\partial_z)$ with the geometric diffusion term in (28) finally yields the Večeř reduction (7): one finds that in (28)

$$z^2\partial_z^2 \mapsto \exp(-\tau L)z^2\partial_z^2\exp(\tau L) = (z-\Delta_X(\tau))^2\partial_z^2 \tag{32}$$

because of the definition (29), giving the relevant geometric diffusion term in (7). That is one can show

$$\begin{aligned}
\left(\partial_\tau - \left(L+\frac{1}{2}\sigma^2z^2\partial_z^2\right)\right)h(z,\tau) &= \left(\partial_\tau - \left(L+\frac{1}{2}\sigma^2z^2\partial_z^2\right)\right)\exp(\tau L\partial_z)g(z,\tau)\\
&= \exp(\tau L\partial_z)\exp(-\tau L\partial_z)\left(\partial_\tau - \left(L+\frac{1}{2}\sigma^2z^2\partial_z^2\right)\right)\exp(\tau L\partial_z)g(z,\tau)\\
&= \exp(\tau L\partial_z)\left(\partial_\tau - \frac{1}{2}\sigma^2(z-\Delta_X(\tau))^2\partial_z^2\right)g(z,\tau), \qquad (33)
\end{aligned}$$

which is equivalent to Večeř's equation (7). Finally, from (26) and (29), one finds the Asian call option boundary data $v(x,y,0) = (y/T-K)^+$ requires $h(z,0) = z^+ = g(z,0)$, the latter of which is (8).

5. The Novel Symmetry and C^∞ Options

We now turn attention to generating a novel reduction of the Asian equation by considering *all* time invariant and homogeneous point reductions of the Asian equation (21), i.e. by setting $\alpha = u_0 = 0$ in (23) but leaving all other parameters there arbitrary. Importantly we consider the new symmetry expressed in (23) by allowing $\gamma \neq 0$. We again integrate the associated symmetry invariance equation (25) to find a more interesting analog of (26), namely the general solution of (25) with these parameters, which can be represented in the form

$$v(\tau,x,y) = \frac{\exp\left[\sigma^{-2}\left(x\frac{Q'(y)}{Q(y)} - \frac{2\Xi}{\Lambda}\operatorname{arctanh}\left(\frac{Q'(y)}{\Lambda}\right)\right)\right]}{(Q(y))^{r\sigma^{-2}}}F\left(\tau,\frac{x}{Q(y)}\right). \tag{34}$$

In (34) we have introduced

$$\Lambda = \sqrt{\beta^2-4\sigma^2\gamma\delta}, \quad \Xi = r\beta+\sigma^2\epsilon, \quad Q(y) = \delta+\beta y+\sigma^2\gamma y^2, \tag{35}$$

and $F = F(\tau,z)$ as any sufficiently smooth function of its arguments. One confirms that there are solutions $v(x,y,\tau)$ of the Asian equation (21) of the form (34) by noting that substitution of (34) into (21) determines a differential equation for $F = F(\tau,z)$, namely

$$\partial_\tau F = \left(-r+rz\partial_z+\frac{1}{2}\sigma^2z^2\partial_z^2\right)F + \sigma^{-2}\left(\Xi z - \frac{1}{2}\Lambda^2z^2\right)F. \tag{36}$$

We emphasize that, up to simple changes of variable, (36) gives the most general time invariant, homogeneous point reduction of the Asian equation (21). In particular one recovers the Večeř reduction (27) by a) choosing the parameters in (35) as per (24), but also by b) using the independent variable

$$z' = \frac{1}{z} = \frac{Q(y)}{x} = \frac{\delta+\beta y+\sigma^2\gamma y^2}{x} = \frac{-KT+y}{x} \tag{37}$$

in (34) rather than $z = x/Q(y)$. On the one hand, the choice $z = x/Q(y)$ is evidently more natural from the point of view of making contact with the Black-Scholes-Merton (BSM) equation: We note that (36) becomes the BSM equation as Ξ and Λ tend to 0, and in this limit one finds that (34) gives

$$v(\tau,x,y) = \frac{\exp\left(-2\sigma^{-2}\frac{x}{\beta'-y}\right)}{(\beta'-y)^{2r\sigma^{-2}}}F\left(\tau,\frac{x}{(\beta'-y)^2}\right). \tag{38}$$

On the other hand we construct a new, C^∞ put-like option with expiration T and "cut-off" price K by choosing $\beta' = KT$ in (38) and using the new independent variable w given by

$$\frac{KT}{\sqrt{K}}w = \frac{1}{\sqrt{z}} = \frac{\beta'-y}{\sqrt{x}} = \frac{KT-y}{\sqrt{x}}: \tag{39}$$

Defining $G = G(\tau, w)$ through

$$F(\tau, z) = CG\left(t, \frac{\sqrt{K}}{KT}\frac{1}{\sqrt{z}}\right) = CG\left(\tau, \frac{1 - y/KT}{\sqrt{x/K}}\right) \equiv CG(\tau, w), \tag{40}$$

where

$$G(0, w) = \theta^+(w) = \theta^+\left(\frac{1 - y/KT}{\sqrt{x/K}}\right) = \theta^+(1 - y/KT), \tag{41}$$

we get the desired put like payoff. (In (41), θ^+ indicates the Heaviside or unit step function, which is supported with value 1 only for positive argument.) In summary, using (38) and (40) (with a certain useful choice of C), one finds that an Asian-like option with payoff

$$v(0, x, y) = \frac{K}{2}\frac{T\sigma^2}{1 - rT}\frac{\exp\left(-\frac{2}{T\sigma^2}\frac{x/K}{1-y/KT}\right)}{(1 - y/KT)^{2r\sigma^{-2}}}\theta^+\left(1 - y/KT\right) \tag{42}$$

has values at other times τ remaining until expiration given by

$$v(\tau, x, y) = \frac{K}{2}\frac{T\sigma^2}{1 - rT}\frac{\exp\left(-\frac{2}{T\sigma^2}\frac{x/K}{1-y/KT}\right)}{(1 - y/KT)^{2r\sigma^{-2}}}G\left(\tau, \frac{1 - y/KT}{\sqrt{x/K}}\right), \tag{43}$$

where $G = G(\tau, w)$ satisfies the initial value problem

$$\partial_\tau G = \left(-r + \left(\frac{3}{8}\sigma^2 - \frac{r}{2}\right)w\partial_w + \frac{1}{8}\sigma^2 w^2 \partial_w^2\right)G, \quad G(0, w) = \theta^+(w). \tag{44}$$

With the simple data indicated, the BSM variant (44) has a simple solution $G(\tau, w) = e^{-r\tau}\theta^+(w)$. Thus we find that an Asian-like option with market dependent payoff given by (42) has values at other times τ before expiration given by

$$v(\tau, x, y) = e^{-r\tau}\frac{K}{2}\frac{T\sigma^2}{1 - rT}\frac{\exp\left(-\frac{2}{T\sigma^2}\frac{x/K}{1-y/KT}\right)}{(1 - y/KT)^{2r\sigma^{-2}}}\theta^+\left(1 - y/KT\right), \tag{45}$$

i.e. its time evolution is simply that of the time value of the relevant bond. Here we evidently restrict to expirations T short enough so that $rT < 1$. The choice of

$$C = \frac{K}{2}\frac{T\sigma^2}{1 - rT} \tag{46}$$

in (40) requiring the restriction $rT < 1$, and which leads to that factor showing in (45), was selected so that

$$\left.\frac{d}{dx}v(\tau, x, Tx)\right|_{x,\tau=0} = -1, \tag{47}$$

which then mimics the behavior of a standard European put in the regime of maximum payoff. To see this behavior explicitly, while noting other important differences in this payoff structure from that of a standard European put, see Figure 1. There we plot in green $v(\tau, x, Tx)$ versus the underlyings indicated by x and $y = xT$ for the market and claim parameter values $T\sigma^2 = 0.45$, $rT = 0.40$, and $K = 80$. In such case one finds that

$$v(0, 0, 0) = \frac{K}{2}\frac{T\sigma^2}{1 - rT} = \frac{80}{2}\frac{.45}{1 - .4} = 30, \tag{48}$$

and the option is similar in value to that of a European put with strike 30 when such may pay maximally. In order to compare, in Figure 1 we also plot the payoff of such a European put in blue. There we see that for values of the underlyings indicated by x and $y = xT$ above 0 but below the "cut-off" value $k = 80$, at expiration the value of the new option exceeds that of the standard European put with strike 30. On the other hand, it should also be clear that, for values of the underlyings x and $y = xT$ below 80, the value at expiration of the new option is exceeded by that of the standard European put with strike 80 . In Figure 1 we plot $v(\tau, x, Tx)$ versus x for values of time $\tau = 1T, .75T, .5T, .25T$ until expiration, as well as at expiration, $\tau = 0$. Curves plotted more boldly correspond to times nearer expiration. As we have mentioned previously, and as should be clear from either the formula (45) or Figure 1, this analogue of the European put is C^∞ in all values of its underlyings, and this at all times, up to and including expiration.

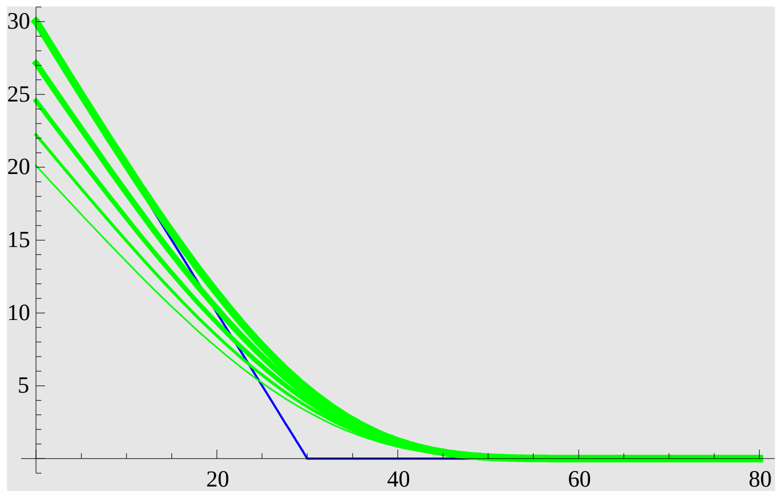

Figure 1. $v(\tau, x, xT)$ versus underlying values x and $y = xT$ for times $\tau = 1T, 0.75T, 0.50T, 0.25T$, and $\tau = 0$ until expiration.

Finally we also note that as per (45) one finds

$$\Delta \equiv \frac{d}{dx} v(\tau, x, y) < 0, \tag{49}$$

$$\Gamma \equiv \frac{d^2}{dx^2} v(\tau, x, y) > 0, \tag{50}$$

for all values of the underlyings x, y. (49) shows that to hedge the short position in this put-like option, one must short the underlying stock in the amount indicated, and (50) indicates that hedging this short position will always be stable under sudden movements of the main underlying x. See for example [12, p. 280] to see that, like the European, replicating any Asian-like option is associated with Δ-hedging in the form indicated in (49).

6. Conclusion

We have constructed the complete maximal 6-dimensional group of point symmetries of the Asian option PDE, thereby extending the known 5-dimensional space of symmetries. The generator of the new symmetry has the unique feature that is depends explicitly on market model parameters.

We have also provided a brief tutorial on the meaning and use of Lie point symmetries. Beyond that we have demonstrated how group invariant solutions of the Asian PDE (21) can be used to generate a novel claim on a stock and its accumulated value that, within the indicated simple market model, nevertheless has $1+1$ dimensional dynamics similar to that arising from the usual Asian option payoff. For this demonstration, we have used principally the new found symmetry, which is generated by the infinitesimal symmetry generator associated with the parameter γ in the complete vector space of symmetry generators demonstrated in (23). The put-like and market-dependent payoff generated by this new symmetry was shown to have precisely the dynamics of the relevant bond, and also to be C^∞ and have uniform convexity in the financial variable, including at expiration and at the (relevant notion of) strike price. Thus we present it as an option which should have particularly favorable hedging properties for all values of its contractual underlyings and at all times during the contract, including up to expiration.

We emphasize that (36) gives the most general time invariant, homogeneous point reduction of the Asian equation (21), which includes then all known reductions as special cases. Importantly (36) is a simple variant of the Black-Scholes-Merton equation, and reduces to such precisely for many nontrivial payoff choices, including the one we have described here in some detail. "Nontrivial" in this context means that a claim is written on both the final value of the underlying stock as well as its time average. All such nontrivial claims whose evolutions are described by the Black-Scholes-Merton equation have then closed form solution algorithms; computing the time evolving values of any such option (with benign boundary conditions) reduces to integrating the contractual claim against the relevant BSM Green's function.

Finally, in the references and appendices we have summarized and directed the reader to the fundamental and useful theorems of Lie's method of generating point symmetries. In that regard, we emphasize that, according to the relevant theorems, the complete classical symmetry group of the Asian option PDE is determined by including the new market dependent symmetry found here with the other five symmetries already known. This follows because these theorems indicate that in order to develop the complete space of classical point symmetries it is sufficient to determine the action of candidate vector fields of symmetries on the n-th order jet space (see theorems A.1 and A.2) of dependent and independent variables, where n is the order of the relevant differential equation; if this action preserves the manifold defined by the equation and its extension to the relevant jet space, then the vector field generates a point symmetry, and conversely. Thus there are no more generators of the point symmetry group of the Asian equation other than the six that we have found by Lie's method. In implementing this complex program of analysis, our use of computer-aided computations was essential. Thus we emphasize the importance of such computer-aided computations, specifically the use of software designed for generating Lie symmetries (c.f. [7]).

Appendix A. The Symmetry Analysis Method

Performing a symmetry analysis on a system of nonlinear equations is widely regarded as the best way to construct exact solutions of the system. The goal of this method is to construct all point transformations of independent variables x^i and dependent variables ϕ^j represented by $(x^i, \phi^j) \to (\tilde{x}^i(x^k, \phi^{k'}), \tilde{\phi}^j(x^l, \phi^{l'}))$ that leave a specified equation unchanged. We summarize the Lie method using the notation in [11]. Other less technical references for the symmetry analysis technique are [5], [7] and [13].

Consider a system of partial differential equations given by

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l, \tag{51}$$

with $x = (x^1, \ldots, x^p)$ the set of independent variables and $u = (u^1, \ldots, u^q)$ the set of dependent variables where $1, \ldots, q$ are indices for the set of all partial derivatives of u up to order n. For $u = f(x)$, with $f : \mathbb{R}^p \to \mathbb{R}^q$ and components f^i, $i = 1 \ldots q$, we define the n-th prolongation of f to be a mapping

$$\mathrm{pr}^{(n)} f : \mathbb{R}^p \to U^{(n)}, \tag{52}$$

given by $\mathrm{pr}^{(n)} f = u^{(n)} = \{u_J = \partial_J f\}$ where J is a multi-index allowing for the development of all possible derivatives $\partial_J f$ of f of order n, and where $U^{(n)}$ is the domain of the u_J's. In words, the prolongation map lists all derivatives of u up to a given order. For example if we consider $u = f(x, y)$, we can compute

$$\mathrm{pr}^{(2)} f(x, y) = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}). \tag{53}$$

The space $\mathbb{R}^p \times U^{(n)}$ is called the n-th order jet space of $\mathbb{R}^p \times U$, where U denotes the domain of u. The fundamental idea behind the method of symmetry analysis is to view Δ_ν as a map from the n-th order jet space into $\mathbb{R}^l$, and then to study the subvariety

$$\mathcal{L}_\Delta = \{(x, u^{(n)}) \in \mathbb{R}^p \times U^{(n)} | \Delta(x, u^{(n)}) = 0\}. \tag{54}$$

Now let $M \subset \mathbb{R}^p \times U^{(n)}$ be open. A symmetry group of Δ_ν is a local group of transformations G acting on an open subset M such that when $(x, u^{(n)}) = (x, \mathrm{pr}^{(n)} f)$ is in the subvariety $\mathcal{L}_\Delta$ of (54), then so is $(\tilde{x}, \tilde{u}^{(n)}) = g \cdot (x, \mathrm{pr}^{(n)} f)$, for all $g \in G$ for which $g \cdot (x, \mathrm{pr}^{(n)} f)$ is defined. The difficulty here is deciding how the group element g acts on each component of $\mathrm{pr}^{(n)} f$ given only its action on $\mathrm{pr}^{(0)} f = f$, specifically doing so in way that is consistent with the chain rule. In turn this is facilitated by finding a local rather than global notion of g's action on the zeroth order jet space, i.e. on the space of only the dependent and independent variables. (This is called the projection of g onto this space.) We do this as follows.

Let X be a vector field on only the space of dependent and independent variables $\mathbb{R}^p \times U$. Hence $X : \mathbb{R}^p \times U \to \mathbb{R}^p \times U$. The "infinitesimal" action $(x, u) \mapsto X(x, u)$ begets a local but finite action via exponentiation, as in $(x, u) \mapsto \exp(\epsilon X)(x, u)$, the latter defined for ϵ small enough via the Taylor series of the exponential. Here then we are thinking of X as an infinitesimal generator of a symmetry $g \in G$, the latter restricted to action on the zeroth order jet space. We extend X from a vector field on $\mathbb{R}^p \times U$ to one on the associated n-th order jet space $\mathbb{R}^p \times U^{(n)}$, the latter vector field denoted by $\mathrm{pr}^{(n)} X$, as follows:

Theorem 1. *If*

$$X = \sum_i \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_a \phi_a \frac{\partial}{\partial u^a}, \tag{55}$$

then X has prolongation

$$\mathrm{pr}^{(n)} X = X + \sum_{a,J} \phi_a^J(x, u^{(n)}) \frac{\partial}{\partial u_J^a}, \tag{56}$$

where

$$\phi_a^J(x, u^{(n)}) = D_J\,(\phi_a - \xi^a u_i^a) + \sum_i \xi^i \partial_i u_J^a, \tag{57}$$

and subscripts on u *indicate partial derivatives and* $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$ *is the* J*-th total derivative (with respect to the independent variables* x*). In (56) the sum over multi-index* J *is such as to develop all* n*-th order derivatives.*

Define the Jacobi matrix of Δ_ν to be

$$J_{\Delta_\nu}(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u_J^a} \right) \tag{58}$$

and say Δ_ν is of maximal rank if the rank of J_{Δ_ν} is l. The following may be called the fundamental theorem of the Lie method.

Theorem 2. *Let* Δ_ν *as in (51) be a system of differential equations of maximal rank. If* G *is a local group of transformations acting on* M *and*

$$\mathrm{pr}^{(n)} X[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \ldots, l, \tag{59}$$

whenever $\Delta_\nu = 0$*, for every infinitesimal generator* X *of* G*, then* G *is a symmetry group of* Δ_ν*.*

This theorem gives sufficient conditions for a vector field X to be an infinitesimal generator of the (point) symmetry group G for the system Δ_ν. The "infinitesimal criterion" (59) is also necessary, so that all such generators are described by (59), when, in addition to our equations having maximal rank, they are also "locally solvable". This occurs when, for example, the system is analytic and can be written in Cauchy-Kovalevskaya form, an instance of this form being the evolutionary form of (21).

We use these two theorems to calculate all infinitesimals generators of the symmetry group G of a maximal rank, locally solvable system of equations, such as (21). We may then exponentiate the infinitesimal generators to obtain the symmetry group of the system. Finally, we apply these symmetries to known and usually simple solutions of the system to obtain new and hopefully more interesting solutions. For a concrete example of this method, we refer the reader to a calculation of the point symmetry group of the heat equation as in [11].

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