Math 112 Sections 11-15: Exam 1

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Name:

Instructions: Problems 1-12 are multiple choice. Mark the correct answer on your bubble sheet. For problems 13-18, write your answer in the space provided. You must show your work for full credit. No book, notes, or calculator are allowed.

Problems 1-12 (4 pts each) (The bolded capital letters are the correct solutions).

1) The numbers e and π are elements of which set(s)?

A. $\mathbb{R}$ b. $\mathbb{Q}$ c. $\mathbb{Z}$ d. (a) and (b)
e. all of the above f. None of the above

Solution: As stated in lecture and the review sessions, e and π are irrational numbers. Stated in mathematical terms, $\{e, \pi\} \in \mathbb{R}\backslash\mathbb{Q}$. Thus they are not integers ($\mathbb{Z}$) or rationals ($\mathbb{Q}$). They are real numbers by their respective definitions. Hence $a.$ is the correct solution.

□

2) If $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a < b$, then $ca < cb$ iff

a. $c = 0$ b. $c = 1$ **C.** $c > 0$ d. 0
e. $c < 0$ f. None of the above

Solution: The proof of this statement is as follows:
($\longleftarrow$) Suppose that $a < b$ for a, b real numbers. If $c > 0$, then $ac < bc$ by axiom 6(iv).[1]

($\longrightarrow$) Suppose that $a < b$ for a, b real numbers, and $ac < bc$. Then $c \neq 0$, since this would imply that $0 < 0$ which is false. Moreover c cannot be negative. If this were so, divide both sides of $ac < bc$ by c, and we find $a > b$ which contradicts our hypothesis. Thus $c > 0$, and we have proven the theorem for choice c).

Recall this property was stated in lecture, and there was a very similar homework problem in section 1.1. Moreover, this theorem is very important, since it is used in every δ, ϵ, proof.

□

3) The domain of $f(x) = \sqrt{x^2 - 2}$ is[2]

a. $(-\sqrt{2}, \sqrt{2})$ b. $[-\sqrt{2}, \sqrt{2}]$ **C.** $\mathbb{R}\backslash(-\sqrt{2}, \sqrt{2})$
d. $\mathbb{R}\backslash[-\sqrt{2}, \sqrt{2}]$ e. $\mathbb{R}$ f. none of the above

Solution: Note that $f(x)$ is not defined when $x^2 - 2 < 0$ or $-\sqrt{2} < x < \sqrt{2}$. Thus $f(x)$ is defined on $\mathbb{R}\backslash(-\sqrt{2}, \sqrt{2})$, which is choice c).

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[1] Note that you did not have to remember the specific axiom for this part of the proof, but rather only realize that multiplication by of a positive number into an inequality, preserves the inequality.

[2] The "\" symbol for two sets A and B defines a new set $A\backslash B$ given by the elements of A less the elements of B.

4) Let $f(x) = e^x$ and $g(x) = \cos x$. Then $(f \circ g)(x)$ is

A. $e^{\cos x}$ b. $\cos e^x$ c. $e^x \cos x$
d. $e^x - \cos x$ e. e^x f. None of the above

Solution: By definition of function composition we have

$$(f \circ g)(x) = f(g(x)) = e^{\cos x}$$

□

5) If $f(x) = 3x + 2$ then $f\left(1 - \frac{1}{f(x)}\right)$ is

a. x **B.** $\frac{15x+7}{3x+2}$ c. $\frac{x}{x+2}$

d. $\frac{7x}{3x+2}$ e. $\frac{5x-3}{x+2}$ f. None of the above

Solution: This problem was taken directly off of the homework.

□

6) Let $f(x)$ and $g(x)$ be two functions. We say g is the inverse function of f iff $(f \circ g)(x) = (g \circ f)(x) = h(x)$, where $h(x)$ is given by

a. 1 b. 0 **C.** x d. None of the above

Choice c) is correct by definition of inverse function.

□

7) The equation $5^x 4^{x+1} = 20^x$ has which of the following real solutions

a. 0 b. 1 c. $\log_5(4)$
d. $\frac{\log_5 4}{\log_5 20 - \log_5 4 - 1}$ **E.** No solution

Solution: This is problem 3d from section 1.8. Note that the question asks for *real* solutions. Choice d) is not a real number, even though it solves the equation. Hence, e) is the correct choice.

□

8) What is the exact value of $\arccos(\sqrt{3}/2)$?

a. 0 b. 1 **C.** $\pi/6$ d. $\pi/2$
e. $\pi/3$ f. None of the above

Solution: This is problem 1b) from the homework for 1.9. Construct a right triangle with opposite side $\sqrt{3}$ and hypotenuse 2 to see why the answer is $\pi/6$.

□

9) Calculate: $\lim_{x\to 1}(x^2-1)$

A. 0 b. 1 c. 2 d. 3
e. 4 f. Does not exist

Applying theorems of 2.3, we have

$$\lim_{x\to 1}(x^2-1) = \lim_{x\to 1} x^2 - \lim_{x\to 1} 1 = 1 - 1 = 0$$

□

10) Calculate: $\lim_{x\to 0}\cosh x$

a. 0 **B.** 1 c. e d. $2e$
e. e^2 f. does not exist

Solution: From the definition of $\cosh x$, we have

$$\lim_{x\to 0}\cosh x = \frac{1}{2}\left(\lim_{x\to 0} e^x + \lim_{x\to 0} e^{-x}\right) = 1$$

□

11) Calculate: $\lim_{x\to 1}\frac{e}{e-1}\sinh x$

a. 0 b. $\frac{e}{e-1}$ **C.** $\frac{e+1}{2}$
d. $\frac{e-1}{2}$ e. $e+1$ f. $e-1$

Solution: From the definition of $\sinh(x)$, we have

$$\begin{aligned}\lim_{x\to 1}\frac{e}{e-1}\sinh(x) &= \frac{e}{2(e-1)}\lim_{x\to 1}(e^x - e^{-x}) \\ &= \frac{e(e-e^{-1})}{2(e-1)} = \frac{e^2-1}{2(e-1)} = \frac{e+1}{2}\end{aligned}$$

□

12) Calculate: $\lim_{x\to 0}\frac{\sin x}{x}$

a. 0 **B.** 1 c. -1
d. Does not exist e. None of the above

Solution: This limit was discussed in section 2.2. If you forgot its value, one line of reasoning to establish the correct limit is the following: Take x to be slightly less than zero (a small negative number). Then both the numerator and denominator of the limiting function are negative, and hence the limit is positive. Similarly, if we consider small positive x, then both the numerator and denominator are positive. Hence the overall limit must be zero or positive if it exists. Moreover, near 0, $\sin x \approx x$ (the sine function looks very much like the identity function) from which we would conclude the correct solution.

□

13a) (4pts) Give the equation of the line in slope-intercept form that passes through the points (2,0) and (3,4).

Solution: We note that the slope of this line is $m = (4-0)/(3-2) = 4$. Thus is has the form

$$y = 4x + b$$

Substituting $(2,0)$ into this equation we find $b = -8$. Thus the line is given by

$$y = 4x - 8$$

□

b) (4pts) Give the equation of the line in slope-intercept form that is perpendicular to the line in part a) and passes through the point (4,5).
Solution:
The line perpendicular to the line in a) has slope $m = -1/4$, and equation

$$y = -\frac{1}{4}x + b$$

Substituting in the point, we have

$$5 = -1 + b \rightarrow b = 6$$

Hence the line has equation

$$y = -\frac{1}{4}x + 6$$

□

14) (7pts) A sinusoid $f(x) : \mathbb{R} \to [10, 14]$ has period 12π, and x-intercept $f(1) = 10$. Find a function that models this sinusoid and graph your result.

Solution: A sinusoid has the form

$$f(x) = a\sin(bx + c) + d$$

The amplitude of this sine wave is $a = (14 - 10)/2 = 2$. The vertical shift is $(14 + 10)/2 = 12$. The period determines b by $12\pi = 2\pi/b \to b = 1/6$. Thus the sinusoid is of the form

$$f(x) = 2\sin(1/6x + c) + 12$$

Since $f(1) = 10$, we find

$$10 = 2\sin(1/6+c)+12 \to -1 = \sin(1/6+c) \to -\frac{\pi}{2} = c+1/6 \to c = -1/6-\pi/2$$

Thus the sinusoid is given by the graph of

$$f(x) = 2\sin(1/6x - 1/6 - \pi/2) + 12$$

Graphing the function should be straightforward.

□

15) (7pts) Suppose you sell pest control and discover an apartment with an initial cockroach population of 300. From past experience, you know the growth rate of an arbitrary cockroach population is 20% per day.

i) Find the doubling time in days for the population.
ii) After how many days will the population reach 30,000 cockroaches? (Leave your answer in exact form).

Solution: We model this population with an exponential function $f(t) = Ae^{kt}$ where $k > 0$. Substituting in the initial value and growth rate, our model becomes $f(t) = 300e^{.2t}$. The population doubles when $f(t) = 600$. Solving the relevant equation, we have

$$2 = e^{.2t} \to \ln(2) = .2t \to t = 5\ln(2) \approx 3.47 \text{ days}$$

To find when there are $3 \cdot 10^4$ cockroaches, we compute

$$3 \cdot 10^4 = 300e^{.2t} \to 100 = e^{.2t} \to t = 5\ln(100) \approx 23.0 \text{ days}$$

□

16) (10pts) In the homework you derived the following formula:

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

The function $y = \cosh x$ is not one-to-one, so it has no inverse function. If we restrict the domain of the function to $[0, \infty)$, the resulting function is one-to-one. We define the inverse hyperbolic cosine function by

$$y = \cosh^{-1} x \text{ if and only if } x = \cosh y \text{ and } y \geq 0$$

By the same process that you used in the homework, find a formula for $y = \cosh^{-1} x$.

Hint: Your answer should be similar to the above formula for $\sinh^{-1} x$.

Solution: Interchanging x and y, we seek to solve

$$x = \cosh y = \frac{e^y + e^{-y}}{2} \rightarrow e^y + e^{-y} - 2x = 0$$

Multiplying the previous equation by e^y, we obtain

$$e^{2y} - 2xe^y + 1 = (e^y)^2 - 2x(e^y) + 1 = 0$$

This is a quadratic equation in e^y. Thus by the quadratic formula, we find

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Thus taking the positive part, we have

$$y = f^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$

□

17. Solve the following equations giving all real solutions

a) (4pts) $\cos 2x = 1$

Solution: Graphically, we note that the equation has two solutions on one of its periods $[0, \pi]$, the first is 0, and the second is π. By extension, the solution set to this equation S is all integer multiplies of π given by

$$S = \{\pi n | n \in \mathbb{Z}\}$$

□

b) (4pts) $\sin 2x = -\cos 2x - 2$

Solution: We rearrange this equation to read

$$\sin 2x + \cos 2x = -2$$

We note since both of these functions have minimum -1, if the above equation is ever satisfied, we must find an a such that $\sin 2a = \cos 2a = -1$. Taking inverses, we find from the sine equation that $2a = -\pi/2 \to a = -\pi$. The cosine equation yields $2a = \pi \to a = \pi/2$. Hence, a must be two different values and no solution exists. Thus $\{\emptyset\}$ is the correct solution.

□

18a) (4pts) State the δ, ϵ definition of the limit.
Solution: See your text or the notes.

b) (8pts) Using the δ, ϵ definition of the limit, prove the following

$$\lim_{x \to 3} (x^2 - 3x + 2) = 2$$

Solution: We seek to bound $|f(x) - L|$ by epsilon assuming that $\delta > 0, \epsilon > 0$ and $0 < |x - 2| < \delta$. We thus compute

$$|f(x) - L| = |x^2 - 3x + 2 - 2| = |x^2 - 3x| = |x| \cdot |x - 3| < |x|\delta$$

Near 3, $|x| \approx 3$, hence it is bounded by the constant function 4 ($|x| < 4$) in any neighborhood about $x = 3$. Thus we find

$$|f(x) - L| < |x|\delta < 4\delta$$

Choosing $\delta < \epsilon/4$ we find

$$|f(x) - L| < 4\delta < 4 \cdot (\epsilon/4) = \epsilon$$

Thus we have shown there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$, and thus have completed the proof.

$\square$

Extra Credit (5pts): In the special theory of relativity, the statement "no massive object may have a velocity greater than or equal to the speed of light ($c \approx 1.9 \cdot 10^5$ miles/second) in vacuum" is taken as an axiom based on experimental evidence. A consequence of this postulate due to Einstein is that an object's mass is not an absolute quantity. Instead it is given by

$$m_{rev} = \frac{m_0}{\sqrt{1-(v/c)^2}}$$

where m_0 is the object's "rest mass" as viewed by a stationary observer, v is the velocity of the massive object relative to the same observer, and c is the speed of light.

Compute:

$$\lim_{v\to 0} m_{rev}, \quad \lim_{v\to c/2} m_{rev}, \quad \lim_{v\to c} m_{rev}, \quad \lim_{v\to 2c} m_{rev}$$

If you have a mass of 100(kg), use the above limits to find your mass when you run at velocities $v = 0, v = c/2, v = c$, and $v = 2c$ relative to a stationary observer, and provide a physical interpretation for your results.

Solution: First note you do not need to know *anything* about special relativity to solve this problem. It is in my opinion one of the most interesting applications of the limit. We compute the desired limits:

$$\lim_{v\to 0} \frac{m_0}{\sqrt{1-(v/c)^2}} = m_0$$

$$\lim_{v\to c/2} \frac{m_0}{\sqrt{1-(v/c)^2}} = \frac{m_0}{\sqrt{3/4}} = \frac{2}{\sqrt{3}}m_0 \approx 1.15m_0$$

$$\lim_{v\to c} \frac{m_0}{\sqrt{1-(v/c)^2}} = \infty$$

$$\lim_{v\to 2c} \frac{m_0}{\sqrt{1-(v/c)^2}} = \frac{m_0}{\sqrt{-3}} = -\frac{i}{\sqrt{3}}m_0$$

Thus if you have a mass of 100 (kg) and were moving with a velocity of 0, your mass would be perceived to be 100 (kg) by any other person at rest. If you were running at half the speed of light $c/2$, you would have a perceived mass of 115 (kg) by a person at rest. As your velocity approaches the speed of light c, your mass would approach ∞ as viewed by an observer at rest. If you could travel faster than the speed of light, you would have *imaginary* mass. Such objects are called **tachyons** and are believed to be unphysical.

□