Math 112 Exam 2

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Section 12

Instructions: Answer all questions. Work on scratch paper will not be graded under any circumstance. You may write on the back if you need more space. You will need a four function calculator to solve one of the problems. Get one from the testing center if you forgot your calculator.

1 Limits

(20pts) Mark the correct answer on your bubble sheet.

Find the limit if it exists of the following:

The capitalized letters are the correct responses.

1. $\lim_{x\to 0} \frac{ax^2 - bx}{5x}$

a. $\frac{b}{5}$ B. $-\frac{b}{5}$ c. $\frac{a}{5}$ d. $-\frac{a}{5}$ e. $-\frac{a}{5}$ f. None of the above

2. $\lim_{x\to\infty} (14.3^{-x} + 2)$

a. 14.3 b. 0 c. ∞ d. $-\infty$ E. 2 f. None of the above

3. $\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1}$

a. 0 b. $-\frac{2}{3}$ C. $\frac{2}{3}$ d. $\frac{1}{3}$ e. $-\frac{1}{3}$ f. None of the above

4. $\lim_{x\to 0} \frac{\sin ax}{bx}$

A. $\frac{a}{b}$ b. $\frac{b}{a}$ c. $\frac{a}{2}$ d. b e. 1 f. None of the above

5. $\lim_{x\to 1} \frac{e^{2x} - e^2}{e^x - e}$

a. 0 b. e c. e^2 d. ∞ E. $2e$ f. None of the above

6. $\lim_{x\to 0} \frac{2x}{\tan x}$

A. 2 b. $\frac{1}{2}$ c. 0 d. 1 e. $-\frac{1}{2}$ f. None of the above

7. $\lim_{x\to\infty} \frac{x^3+5}{3x^3-x^2+7}$

a. ∞ b. -∞ c. 0 D. $\frac{1}{3}$ e. 3 f. None of the above

8. $\lim_{x\to\infty} \sin\frac{1}{x^2}$

A. 0 b. -1 c. ∞ d. 1 e. $-\infty$ f. None of the above

9. $\lim_{x\to\infty} \sqrt{x^2+ax+b} - x$

a. a B. $\frac{a}{2}$ c. $\frac{a}{b}$ d. b e. 0 f. None of the above

10. $\lim_{x\to\infty} \frac{1}{1-e^{ax}\tan(bx)}$

a. a b. $\frac{a}{2}$ c. ∞ d. $-\infty$ e. 0 F. None of the above

2 Derivatives

(24pts) Mark the correct answer on your bubble sheet.

Find the derivative of the following:

11. $f(x) = 3x^7 + \pi$

a. $3x^6$ b. $21x^7$ C. $21x^6$ d. $\frac{3}{7}x^6$ e. $\frac{3}{7}x^7$ f. None of the above

12. $f(x) = e^x \arctan x$

a. $\frac{e^x}{1-x^2} + e^x \arctan x$ b. $\frac{e^x}{\sqrt{1-x^2}} + e^x \arctan x$ c. $\frac{-e^x}{\sqrt{1-x^2}} + e^x \arctan x$ d. $\frac{e^x}{1+x^2} + \arctan x$
e. $e^x \arctan x$ F. None of the above

13. $f(x) = \dfrac{x+1}{e^x+1}$

A. $\frac{1-xe^x}{(e^x+1)^2}$ b. $\frac{1-xe^x}{e^x+1}$ c. $\frac{1+xe^x}{(e^x+1)^2}$ d. $\frac{1-e^x}{(e^x+1)^2}$ e. $\frac{1-xe^x}{(e^x-1)^2}$ f. None of the above

14. $f(x) = \sin x \cos x$

a. -1 b. $\cos^2 2x - \sin^2 2x$ c. $2\cos^2 x - 2\sin^2 x$ d. 1
E. $\cos^2 x - \sin^2 x$ f. None of the above

15. $f(x) = (1 - \tan x)^3$

a. $-3\sec^2 x(1-\tan x)^3$ b. $-\sec^2 x(1-\tan x)^2$ c. $-3(1-\tan x)^2$ d. $3\sec^2 x(1-\tan x)^2$
E. $-3\sec^2 x(1-\tan x)^2$ f. None of the above

16. $f(x) = \cos(x^2)$
a. $-x\sin(x^2)$ b. $\sin(x^2)$ c. $-\sin(x^2)$ d. $-x^2\sin(x^2)$ E. $-2x\sin(x^2)$

f. None of the above

17. $f(x) = \sin(e^{-x/5})$

a. $\frac{1}{5}e^{-\frac{x}{5}}\cos(e^{-\frac{x}{5}})$ b. $e^{-\frac{x}{5}}\cos(e^{-\frac{x}{5}})$ c. $-\frac{1}{5}\cos(e^{-\frac{x}{5}})$
d. $-\frac{1}{5}e^{-\frac{x}{5}}\sin(e^{-\frac{x}{5}})$ e. $-\frac{x}{5}e^{-\frac{x}{5}}\cos(e^{-\frac{x}{5}})$ F. None of the above

18. $f(x) = \frac{3^{2x}}{1+\cos 3x}$

a. $\frac{2\ln 3(3^{2x})(1+\cos 3x)+3(3^{2x})\sin 3x}{1+\cos 3x}$
b. $\frac{\ln 3(3^{2x})(1+\cos 3x)+3(3^{2x})\sin 3x}{(1+\cos 3x)^2}$
c. $\frac{2(3^{2x})(1+\cos 3x)+3(3^{2x})\sin 3x}{(1+\cos 3x)^2}$
d. $\frac{2\ln 3(3^{2x})(1+\cos 3x)-3(3^{2x})\sin 3x}{(1+\cos 3x)^2}$
e. $\frac{2\ln 3(3^{2x})(1+\cos 3x)+3(3^{2x})\sin 3x}{(1+\cos 3x)^2}$
F. None of the above

3 Free Response

Write your answer in the space provided. Show all of your work.

19.
Differentiate the following function: $f(x) = \frac{\tan x}{1-\sec^2(\cos x)} + \log_b(3x)e^{ax} - \text{arccsc}(x) + \cosh x$ (5pts)

Solution:
We differentiate each term and note that the solution is the sum of the derivatives of the four terms.

$$\frac{d}{dx}\left(\frac{\tan x}{1-\sec^2(\cos x)}\right) = \frac{(1-\sec^2(\cos x))\sec^2 x - 2\tan x \sec(\cos(x))^2 \sin x \tan(\cos x)}{(1-\sec^2(\cos x))^2}$$

$$\frac{d}{dx}\log_b(3x)e^{ax} = \frac{e^{ax}(1+ax\ln(3x))}{x\ln b}$$

$$-\frac{d}{dx}\text{arc}\csc x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cosh x = \sinh x$$

□

20. Find a formula for the n-th derivative for the function $f(x) = x^m$ where $m \in \mathbb{Z}$ (5pts).

See homework for solution.

□

21. Assume that the equation $x^2 - 2xy - y^2 = 0$ implicitly defines y as a function of x. Find the slope of the tangent line to the graph of the equation at the point $(2, 2\sqrt{2} - 2)$. (6pts)

Solution:
Differentiating implicitly we have

$$2x - 2x\frac{dy}{dx} - 2y - 2y\frac{dy}{dx} = 0$$

Solving for dy/dx we find

$$\frac{dy}{dx}(x + y) = x - y \rightarrow \frac{dy}{dx} = \frac{x - y}{x + y}$$

Substituting in the point we find

$$\frac{dy}{dx} = \frac{2 - \sqrt{2}}{\sqrt{2}}$$

□

22. Define a function

$$f(x) = \frac{\tan 4x}{\sin 2x}$$

Can $f(x)$ be defined so that it is continuous? If not, why not? If so, how so, and what is the defined value? (6pts)

Hint: $\cos(s+t) = \cos s \cos t - \sin s \sin t$ and $\sin(s+t) = \sin s \cos t + \cos s \sin t$

Solution:
If make the definition

$$f\left(\frac{\pi n}{2}\right) = 2 \quad \text{for} \quad n \in \mathbb{Z}$$

then f would be continuous. We need to do this because f is not defined at these points, but limits to two at each of them.

□

This question was thrown out because I believe it was ambiguously stated.

23.
a) State the Mean Value Theorem (3pts)

Solution:
See Text or notes.

□

b) Use Rolle's Theorem or the Mean Value Theorem to show that the function $f(x) = 4 - x - 6x^3$ has at most one real zero. (4pts)

Solution:
Since $f(0) = 4$ and $f(1) = -3$, we note by the intermediate value theorem that $f(x)$ has as least one real zero. Now consider $f'(x) = -18x^2 - 1$. If there are two zeros, z_1 and z_2, then Rolle's theorem guarantees that there is a point $c \in (z_1, z_2)$ such that $f'(c) = 0$. Thus we see $-18c^2 - 1 = 0$ which implies $c = \pm i/\sqrt{18}$. Since $c \in \mathbb{R}$ this gives a contradiction. Thus $f(x)$ cannot have two real zeros. Finally, we note that $f'(x) < 0$ for all x. Since the graph is monotonically decreasing, it can not have three zeros. Thus completes the proof.

□

24.
a) State the definition of the derivative of a function f at a point c. (3pts)

See Text.

$\square$

b) The function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$ and $f'(0) = 0$. Prove this using the definition of the derivative. (6pts)

Solution:
From the definition of the derivative we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 \sin[1/(x+h)] - x^2 \sin(1/x)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h}\left(- x^2 \sin\left(\frac{1}{x}\right) + h^2 \sin\left(\frac{1}{h+x}\right) + 2hx \sin\left(\frac{1}{h+x}\right) + x^2 \sin\left(\frac{1}{h+x}\right)\right)$$
$$= 2x \sin\left(\frac{1}{x}\right) + x^2 \lim_{h \to 0} \frac{[\sin(1/(x+h)) - \sin(1/x)]}{h}$$
$$= 2x \sin\left(\frac{1}{x}\right) + x^2 \frac{d}{dx} \sin(1/x) = 2x \sin\left(\frac{1}{x}\right) - \cos(1/x)$$

Thus we see that $f(x)$ is actually not differentiable at $x = 0$.

$\square$

25. (4pts) Use the local linearization technique to find an estimate for the difference in surface area of a sphere of radius 1.00 and a second sphere of radius 1.2. Find the exact value of the sphere of radius 1.2 and determine the error of the linear approximation.

Solution:
The surface area of a sphere is given by $A = 4\pi r^2$. Taking differentials, we find

$$dA = 8\pi dr$$

. In our case $dr = 0.2$. Thus gives $dA \approx 5.03$. Since the initial surface area of the smaller sphere is approximately 12.6, we conclude that the area of the larger sphere is approximately 17.0.

$\square$

26. (10 pts) State on what intervals the following function is increasing, decreasing, concave up, and concave down. Give the location of all of its critical points and inflection points. Sketch the graph of the function based on the above information.

$$f(x) = x + \cos x$$

Solution:
We compute $f'(x) = 1 - \sin x$ and $f''(x) = -\cos x$. Thus we see that function is increasing when $1 - \sin x > 0$ or when $1 > \sin x$. Since this happens whenever $x \neq (2n+1)\pi/2$ for any $n \in \mathbb{Z}$, we conclude that f is increasing on the set $\mathbb{R} - \{(2n+1)\pi/2 | n \in \mathbb{Z}\}$. Moreover, f is never decreasing.

We note f is concave up when $-\cos x > 0$ or $\cos x < 0$. This occurs on the set of intervals $\{n\pi/2 + 3n\pi/2 | n \in \mathbb{Z}\}$ and n is odd. Finally, the inflection points of f occur when $\cos(x) = 0$ which happens for every element of the set $\{(2n+1)\pi/2 | n \in \mathbb{Z}\}$. The following is a graph of the function:

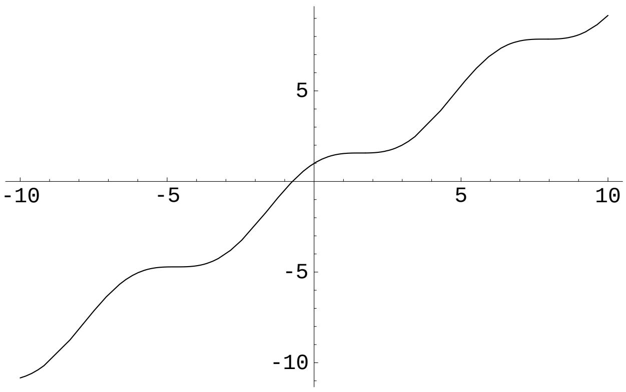

Figure 1: $f(x) = x + \cos x$ for $x \in [-10, 10]$.