Math 112 Exam 1

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Section 12

Answer the following to the best of your ability. Leave all answers in exact form (do not try to compute decimal approximations). Use a **separate** sheet of paper for each question.

Easy Questions:
1. (5 pts) Find the limit of the following if it exists. If it does not exist, explain why.

$$L = \lim_{x \to 0} f(x) \quad \text{where} \quad f(x) = \begin{cases} 1 & \text{when} \quad x > 0 \\ 0 & \text{when} \quad x = 0 \\ -1 & \text{when} \quad x < 0 \end{cases}$$

Solution:
This limit does not exist since

$$\lim_{x \to 0^+} = 1 \neq -1 = \lim_{x \to 0^-}$$

□

2. (10 pts) Find the inverse function of the following and show $(f \circ g)(x) = (g \circ f)(x) = x$:

$$f(x) = e^{3x+8}$$

Solution:
We solve the equation

$$x = e^{3y+8}$$

for y to find

$$g(x) = y = \frac{1}{3}(\ln x - 8)$$

Computing compositions, we have

$$(f \circ g)(x) = \exp(3(1/3(\ln x - 8)) + 8) = \exp(\ln x) = x$$

$$(g \circ f)(x) = \frac{1}{3}(\ln(\exp(3x+8)) - 8) = x$$

$\square$

3a. (6 pts.) Give the equation of the line, in slope-intercept form, that passes through the points (2,0) and (4,5).

Solution:
We compute the slope of the line to be $5/2$. Thus the line has the form

$$y = \frac{5}{2}x + b$$

Substituting in the point $(2, 0)$ we find $0 = 5 + b$. Thus $b = -5$. So the equation for the line is

$$y = \frac{5}{2}x - 5$$

$\square$

3b. (6 pts.) Give the equation of the line, in slope-intercept form, that is perpendicular to the line in 3a and passes through the point (4,5).

Solution:
The line perpendicular to the one in 3 has has negative reciprocal slope. Thus it has the form

$$y = -\frac{2}{5}x + b$$

Substituting the point $(4, 5)$ into the equation we find

$$5 = -\frac{8}{5} + b$$

Thus $b = 33/5$ and we find the desired slope intercept form to be

$$y = -\frac{2}{5}x + \frac{33}{5}$$

$\square$

Medium Questions:
4. (15 pts.) Given the following sinusoidal function $f(x)$, determine its amplitude, period, and phase shift. Then sketch a graph of f labeling the above parts.

$$f(x) = 2\sin\left(\frac{1}{2}x - 2\right) - 1$$

Solution: The amplitude of this sinusoid is 2, the period is 4π, and the phase shift is 4. Your plot should look something like the following:

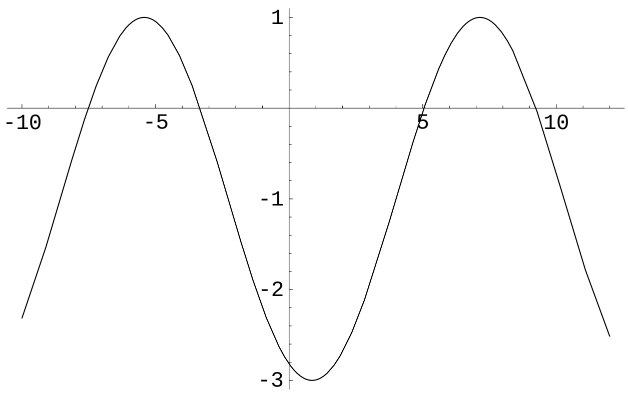

Figure 1: $f(x) = 2\sin\left(\frac{1}{2}x - 2\right) - 1$ for $x \in [-10, 12]$.

□

5. (12 pts.) Suppose you sell pest control and discover an apartment with an initial cockroach population of 300. From past experience, you know the growth rate of an arbitrary cockroach population is 20% per day. After how many days will the population reach 30,000 cockroachs? (Leave your answer in exact form).

Solution: [1]
The exponential model governing the growth of the population is

$$f(x) = 300e^{.2t}$$

We solve the equation

$$30000 = 300e^{.2t}$$

to find that

$$t = \ln(100)/.2 \approx 23.03 \text{ days}$$

□

Do **two** of 6a, 6b, or 6c. Clearly indicate which problem you do not want graded. If it is unclear, the first two will be graded.

6a. (7 pts.) **Prove** the following:
For every $a \in \mathbb{R}$, $a \cdot 0 = 0$.

Solution:
See 1.1 Homework Solutions.

[1] Both continuous and discrete models were excepted. The following solution is for the continuous model.

□

6b. (7 pts.) **Prove** the following:
Let S be a set. If S has a least upper bound, then it is unique.

Solution:
See 1.1 Homework Solutions.

□

6c. (7 pts.) **Prove** the following and then answer the question:
Let $a, b, c \in \mathbb{R}$. If $ac = bc$, and $c \neq 0$, then $a = b$. Why is it necessary to demand $c \neq 0$?

Solution:
See 1.1 Homework Solutions.

□

7. (8 pts.) Using the cosine addition formula and the identity $\sin(\pi/2 - t) = \cos t$, derive the sine subtraction formula

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Solution:
See 1.5 Homework Solutions.

□

Hard Questions:
8a. (5 pts.) State the ϵ, δ definition of the limit.

Solution:
See 2.2 in text.

□

8b. (7 pts.) **Prove** the following using the ϵ, δ definition of the limit:

$$\lim_{x \to 3} (x^2 - 4x + 4) = 1$$

Solution:
Given $\epsilon > 0$ we must find a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 - 4x + 4) - 1| < \epsilon$. Note that

$$|(x^2 - 4x + 4) - 1| = |x^2 - 4x + 3| = |x - 3| \cdot |x - 1| < \delta |x - 1| < 3\delta$$

since $|x - 1| < 3$ when x is in a neighborhood of 3.
Thus if we choose $\delta < \epsilon/3$,

$$|f(x) - L| = |(x^2 - 4x + 4) - 1| < 3\delta < \epsilon$$

which proves the limit is 1 from the definition.

□

9. (12 pts.) Given a parabolic equation of the form

$$f(x) = ax^2 + bx + c \quad \text{for} \quad \{a, b, c\} \in \mathbb{R}$$

derive an expression for the vertex of the parabola. Then give a condition for the vertex being a maximum or minimum of the function.

Solution:
We complete the square on $f(x)$ via the following:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$
$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

If $a > 0$ then the smallest value the graph can take is at x_0 where

$$f(x_0) = \frac{4ac - b^2}{4a} \to x_0 = -\frac{b}{2a}$$

So the vertex has coordinates

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Finally we note that if $a > 0$,

$$\lim_{x \to \pm\infty} f(x) = \infty$$

thus the vertex is a minimum, and if $a < 0$, then

$$\lim_{x \to \pm\infty} f(x) = -\infty$$

thus the vertex is a maximum.

□